FINAL: 180 Minutes

Last Name: Solutions

First Name: ______

RIN: ______

Section: ______

Answer **ALL** questions. You may use **two** double sided $8\frac{1}{2} \times 11$ crib sheets.
You **MUST** show **CORRECT** work (even for multiple choice) to receive full credit.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	6	Total
200	30	30	30	30	30	350

INSTRUCTIONS

1. This is a **closed book** test. No electronics, books, notes, internet, etc.
2. You can have two double sided $8\frac{1}{2} \times 11$ crib-sheets (handed in separately).
3. The test will become available in Submitty at 8am on the test date. Your PDF is due in Submitty by 8am the next day. You have 3 hours to do the exam and 3 additional hours to type your answers and submit a PDF.
4. By submitting the test you attest that the work is entirely your own and you obeyed the time limits of the exam.
5. Your submission *must* be typed PDF. The 3 hour test time for solving the problems must be continuous. The extra 3 hours is to type your answers and explanations: you may take breaks, but not change answers.
6. You *must* show your work for *every* answer immediately after the answer. The format for what you hand in is something like:

   Problem 1
   (1) A
   Because x is even, therefore ...
   (2) B
   Because $\sqrt{2}$ is irrational, therefore ...
   (4) D
   By the law of total expectation, $\mathbb{E}[\mathbf{X}] = \cdots$
   ⋮
   (20) A
   We proved in class that $\ell = n - 1$. Therefore ...
   Problem 2
   ⋮

   – Start each long-answer question on a new page.
   – Some problems may be "easy", so give a short explanation.
   – Some problems may require a detailed reasoning.
   – `3*3+1+3=13` is **not** an explanation. Everyone knows that `3*3+1+3=13`.
   Why this equation? Where do the numbers come from?
7. **If you don't show correct work, you won't get credit.**
8. Be especially careful in the multiple choice.
   – Correct answers get 10 points.
   – Wrong answers or correct answers without correct justification get 0.
9. Submit with plenty of time to spare. A late test won't be accepted.

1 Circle at most one answer per question. 10 points for each correct answer.

(1) "For a constant $c > 0$, $1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n}$, where n is any natural number." Which claim is this?

B

- A $\exists c > 0 : (\exists n \in \mathbb{N} : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.
- B $\exists c > 0 : (\forall n \in \mathbb{N} : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.
- C $\exists n \in \mathbb{N} : (\forall c > 0 : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.
- D $\forall n \in \mathbb{N} : (\exists c > 0 : 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n})$.
- E None of the above.

For a constant c $\rightarrow \exists c>0$

n is any natural # $\rightarrow \forall n \in \mathbb{N}$

$\therefore \exists c : (\forall n : \ldots)$

(2) You will pick a constant $C > 0$ such that no matter which $n \in \mathbb{N}$ I pick, $\sum_{i=1}^{n} i \le Cn$. Which is true?

E

- A You can pick a C satisfying $C \le 10$.
- B You can pick a C satisfying $10 < C < 100$.
- C You can pick a C satisfying $100 < C < 1000$.
- D You can pick a C satisfying $1000 < C$.
- E There is no constant $C > 0$ that you can pick.

$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1) \le Cn$

$\rightarrow C \ge \frac{1}{2}(n+1) \quad \forall n$

$\therefore$ C can't be a constant

(3) $T_1 = 2$ and $T_n = T_{n-1} + 2n$ for $n > 1$. What is T_{100}?

B

No Work No Credit

- A 5050.
- B 10100.
- C 20200.
- D 40400.
- E None of the above.

$T_n = T_{n-1} + 2n$

$T_{n-1} = T_{n-2} + 2(n-1)$

$T_{n-2} = T_{n-3} + 2(n-2)$

$\vdots$

$T_2 = T_1 + 2\cdot 2$

$T_1 = 2\cdot 1$

Σ it. $T_n = 2\cdot 1 + 2\cdot 2 + \cdots 2\cdot n$

$T_n = 2(1+2+\cdots+n)$

$= 2\cdot\frac{n}{2}(n+1)$

$= n(n+1)$

$n = 100 \rightarrow$

$T_n = 100\cdot 101$

$= 10100$.

(4) $T_1 = 1$ and $T_n = n \times T_{n-1}$ for $n > 1$. Which is true?

D

- A $T(n) \in O(n^2)$.
- B $T(n) \in o(2^n)$.
- C $T(n) \in \Theta(2^n)$.
- D $T(n) \in \omega(2^n)$.
- E None of the above.

$T_n = n\, T_{n-1} = n(n-1)\, T_{n-2} = \cdots\; n(n-1)\cdots 1 = n!$

$n! > 2^n$

$\therefore T(n) \in \omega(2^n)$

(5) You divide 2^{2016} candies evenly among 11 kids. How many candies are left over?

Work (mod 11)

D

No Work No Credit

- A 0.
- B 3.
- C 6.
- D 9.
- E None of the above.

$2^1 = 2$

$2^2 = 4$

$2^3 = 8$

$2^4 = 5$

$2^5 = -1$

useful.

$\therefore 2^5 = -1$

$(2^5)^{403} = (-1)^{403} = -1$

$2^{2015} = -1$

$\therefore 2\cdot 2^{2015} = 2^{2016} = 2\cdot -1 = -2 = 9$.

remainder

(6) Estimate the sum $S = \sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.

$\boxed{A}$ $0 < S \le 2$.

$\boxed{B}$ $2 < S \le 2000$.

$\boxed{C}$ $2000 < S \le 20000$.

$\boxed{D}$ $20000 < S \le 200000$.

$\boxed{E}$ None of the above.

E

Harmonic Sum $\to \infty$

$\therefore$ No upper bound.

(7) How many of the numbers $100, 101, 102, \ldots, 999$ do not contain the digit 2?

$\boxed{A}$ 100.

$\boxed{B}$ 504.

$\boxed{C}$ 648.

$\boxed{D}$ 729.

$\boxed{E}$ None of the above.

C

No Work No Credit

n-digits

1's digit: 8 choices $(1, 3, 4, \ldots, 9)$

2^{nd} digit: 9 choices $(0, 1, 3, \ldots, 9)$

$\therefore$ Product rule $8 \times 9 \times \ldots \times 9 = 8 \times 9^{n-1}$

$n = 3 \to 8 \times 9^2 = 8 \times 81 = 648$.

(8) Let S be the sum of the reciprocals of all natural numbers not containing the digit 2. Estimate S.

$\boxed{A}$ $0 < S \le 2$. ×

$\boxed{B}$ $2 < S \le 2000$.

$\boxed{C}$ $2000 < S \le 20000$.

$\boxed{D}$ $20000 < S \le 200000$.

$\boxed{E}$ None of the above.

B

No Work No Credit

$> 1/2$ $\quad > 1/2$ $\quad \therefore S > 2$

$$S = 1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{30} + \frac{1}{31} + \frac{1}{33} + \frac{1}{34} + \frac{1}{35} + \cdots$$

Consider reciprocals of n digits: there are $8 \cdot 9^{n-1}$ such #s. $\frac{1}{x} \le \frac{1}{10^{n-1}}$ $\therefore$ Contribution from reciprocal of n-digits $< \frac{8 \cdot 9^{n-1}}{10^{n-1}}$

$\therefore S < \sum_{n=1}^{\infty} 8 \cdot \frac{9^{n-1}}{10^{n-1}} = 8 \cdot \sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^{n-1} = 8 \cdot \frac{1}{1 - \frac{9}{10}} = 80$ $\therefore \boxed{S < 80}$

(9) Shirts come in 3 colors R, G or B. In how many ways can you distribute shirts to 7 studens?

$\boxed{A}$ $\binom{7}{3}$.

$\boxed{B}$ 7^3.

$\boxed{C}$ 3^7.

$\boxed{D}$ $7!/3!$.

$\boxed{E}$ None of the above.

C

Each Student 3 Choices $\to$ Product rule: $3 \times 3 \times \ldots \times 3 = 3^7$.

(10) Repeat the previous problem if the number of each color shirt is at least ~~three~~ two.

$\boxed{A}$ 570

$\boxed{B}$ 600.

$\boxed{C}$ 630.

$\boxed{D}$ 660.

$\boxed{E}$ None of the above.

C

No Work No Credit.

Anagrams of RRGGBBB and RRGGGBB and RRRGGBB

$$= 3 \cdot \frac{7!}{2!2!3!} = \frac{3 \cdot 7 \times 6 \times 5 \times 4}{2 \cdot 2} = 3 \cdot 7 \cdot 30 = 3 \times 210 = 630.$$

(11) Every vertex in a graph G has degree 1. Which is true?

[A] The graph G must be disconnected. X o—o is connected

[B] The graph G could have 5 vertices. X 5 vertices → degree sum is 5 (odd) → not possible

[C] The graph G must have a cycle. X

[D] The graph G is not possible. X

[E] None of the above

E

(12) You rolled a pair of dice. What are the chances you rolled exactly one 5?

[A] 9/36.

[B] 10/36.

[C] 11/36.

[D] 12/36.

[E] None of the above.

B

Total outcomes 36

Valid outcomes of interest

$\boxed{15}$
25
$\boxed{35}$
45
$\boxed{65}$
$\boxed{51}$
52
$\boxed{53}$
54
56

$10 \rightarrow \frac{10}{36}$

even sum are ~~circled~~ boxed

4 of them

so $P[5 + \text{even}] = \frac{4}{36}$

$P[\text{even}] = \frac{18}{36}$

(13) You rolled a pair of dice. What are the chances you rolled exactly one 5 if the sum is even?

[A] 4/10.

[B] 5/10.

[C] 4/11.

[D] 5/11.

[E] None of the above.

E

No work No credit

$P[5 \mid \text{even}] = \frac{P[5 \text{ and even}]}{P[\text{even}]} = \frac{4/36}{18/36} = \frac{4}{18} = \frac{2}{9}$

(14) Which of the following random variables **X** is not a binomial random variable.

[A] Randomly throw 100 darts at a dart board. **X** is the number of darts hitting the bulls-eye. ✓

[B] Randomly answer 100 5-choice multiple choice questions. **X** is the number of questions correct. ✓

[C] Randomly answer 100 5-choice multiple choice questions. **X** is the number of questions wrong. ✓

[D] 1000 students randomly line up, 500 are boys. **X** is the number of boys in the first 100 students. X Not independent

[E] They are all binomial random variables.

D

(15) A social network (graph) is a *tree* with 20 people. The edges are friendships. Each person randomly picks red or blue. Friends compare to see if they match. What is the expected number of matches.

[A] 4.75.

[B] 5.

[C] 9.5

[D] 10.

[E] None of the above or not enough information.

C

19 edges each edge matches w.p. $\frac{1}{2}$

$X_i = \begin{cases} 1 & \text{if edge Matches} \\ 0 & \text{otherwise} \end{cases}$

$X = X_1 + \cdots X_{19}$

$E[X] = 19\, E[X_1] = 19 \cdot \frac{1}{2} = 9.5$

(16) On BlueToe, your first child is equally likely to be a boy or girl. From then on, the sex of a child is the same as the previous child with probability 2/3. What is the expected number of kids to get a girl?

A 1.5.

B 2.

C 2.5.

D 3.

E None of the above.

C

No work No Credit

$\frac{1}{2}$ B, $\frac{1}{2}$ G

$$E[X] = E[X|G]P[G] + E[X|B]P[B]$$
$$= 1\cdot\tfrac{1}{2} + \big(1 + \underbrace{\text{Exp time to girl}}_{p\ \text{success} = \frac{1}{3}\ \therefore\ \text{Exp Wait} = 3}\big)\cdot\tfrac{1}{2}$$
$$= 1\cdot\tfrac{1}{2} + (1+3)\cdot\tfrac{1}{2} = \tfrac{1}{2} + 2 = 2.5$$

(17) On BlueToe, your first child is equally likely to be a boy or girl. From then on, the sex of a child is the same as the previous child with probability 2/3. What is the expected number of kids to two girls?

A 3.25.

B 4.

C 4.5.

D 5.25.

E None of the above.

C

No Work No Credit

$\frac{1}{2}$ B, $\frac{1}{2}$ G

$$E[X] = E[X|B]\cdot\tfrac{1}{2} + E[X|G]\cdot\tfrac{1}{2} = (1+b)\cdot\tfrac{1}{2} + (1+g)\cdot\tfrac{1}{2}$$

b = Exp wait to 2G given B

g = Exp wait to 1G given G.

Compute b: B: $\frac{2}{3}$ B, $\frac{1}{3}$ G

$$b = (1+b)\cdot\tfrac{2}{3} + (1+g)\cdot\tfrac{1}{3} = 1 + \tfrac{2}{3}b + \tfrac{1}{3}g \rightarrow b = 3 + g$$

Compute g: G: $\frac{1}{3}$ B, $\frac{2}{3}$ G

$$g = (1 + \underbrace{\text{Exp wait to } G}_{3})\cdot\tfrac{1}{3} + 1\cdot\tfrac{2}{3} = \tfrac{4}{3} + \tfrac{2}{3} = \tfrac{6}{3} = 2$$
$$\rightarrow b = 5$$

$$\therefore\ E[X] = (1+5)\cdot\tfrac{1}{2} + (1+2)\cdot\tfrac{1}{2} = \tfrac{9}{2} = 4.5$$

(18) Estimate the number of DFA you can draw with 4 states, q_0, q_1, q_2, q_3. Tinker!

A About a hundred.

B About a thousand.

C About a million.

D About a billion.

E About a trillion.

C

No Work No Credit

Choose YES-States: 2^4 possible subsets of $\{q_0 q_1 q_2 q_3\}$

8 arrows, 4 destinations each $\rightarrow 4^8$ choices.

Total Choices: $2^4 \times 4^8 = 2^4 \times 2^{16} = 2^{20} = (2^{10})^2 \approx (10^3)^2 = 10^6$

(19) Which string can be generated by the CFG $S \to 0|1|SSS$?

A 1111. ✗

B 0000. ✗

C 000111. ✗

D 111000. ✗

E None of the above.

E

No Explanation No Credit

$S \to 0$, $S \to 1$, $S \to SSS \to$ length 3, $\to SSSSS$ ... } all strings of odd length.

(20) If $\mathcal{L}_A$ is decidable, then $\mathcal{L}_B$ is decidable. We know that $\mathcal{L}_B$ is undecidable. Therefore:

A $\mathcal{L}_A$ must be finite.

B $\mathcal{L}_A$ must be infinite.

C $|\mathcal{L}_A| > |\mathcal{L}_B|$.

D $|\mathcal{L}_B| < |\mathcal{L}_A|$.

E None of the above.

B

Correct Explanation Required

L_A is undecidable $\rightarrow$ cannot be finite

all finite languages are decidable

2 Determine the Type of Proof and Prove

Prove that there is a constant $c > 0$ for which, no matter which $n \in N$ you pick,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > c\sqrt{n}.$$

Can make this guess using integration

Tinker $1 > \sqrt{1}$ $\quad 1+\frac{1}{\sqrt{2}} > \sqrt{2}$ $\quad 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}} > \sqrt{3}$

guess $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots \frac{1}{\sqrt{n}} > \sqrt{n}$

$\boxed{c = 1}$

Proof by induction

~~1st~~ Base Case $1 \geq \sqrt{1}$ ✓

Induction Step.

Assume $1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$

Prove $1 + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \sqrt{n+1}$

$1 + \cdots + \frac{1}{\sqrt{n+1}} > \sqrt{n} + \frac{1}{\sqrt{n+1}}$ (induction hypothesis)

to conclude we show $\sqrt{n} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n+1}$.

Lemma $\sqrt{n} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n+1}$

Suppose not (Contradiction).

$\rightarrow \sqrt{n} + \frac{1}{\sqrt{n+1}} < \sqrt{n+1}$

$\downarrow$

$\sqrt{n(n+1)} + 1 < n+1$

$\downarrow$ $\sqrt{n(n+1)} < n$.

$\downarrow$ $n(n+1) < n^2$

$\downarrow$ $n^2 + n < n^2$

$\downarrow$ $n < 0$ $\boxed{\text{FISHY}}$

50% tinker and understand.

80% Made guess and started proof.

100% Proof by induction Correct

Can also be proved by Integration method. Needs Care

3 Product of 5 Consecutive Numbers.

Prove that the product of any 5 consecutive natural numbers is divisible by 5! (e.g. $5! | 3 \times 4 \times 5 \times 6 \times 7$).

Quick Proof $\dfrac{(n+1)(n+2)(n+3)(n+4)(n+5)}{5!} = \dfrac{(n+5)!}{n!\,5!} = \binom{n+5}{5} =$ # ways to choose 5 from $n+5$ must be an integer.

Induction $P(n)$: $n(n+1)(n+2)(n+3)(n+4)$ is divisible by $5!$

We prove $P(n)$ for all $n \geq 1$.

$P(1)$: $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5!$ is div by $5!$ ✓

$P(n) \rightarrow P(n+1)$

Suppose $5! \mid n(n+1)(n+2)(n+3)(n+4)$

Consider $P(n+1)$ $\quad (n+1)(n+2)(n+3)(n+4)(n+5)$

$$= \underbrace{n(n+1)(n+2)(n+3)(n+4)}_{\text{div by } 5!} + \underbrace{5(n+1)(n+2)(n+3)(n+4)}_{?}$$

50% tinker and show understood.
80% Started Proof Correctly
100% Quick Proof or got to here.

Done if $\underbrace{\cancel{4}! \mid (n+1)(n+2)(n+3)(n+4)}$

Same problem: 4! divides product of 4 consecutive numbers.

True by induction if 3! divides product of 3 consecutive

True if 2! divides product of 2 consecutive

true

Since one must be even

4 Expected Waiting Time to All Colors of Starbust.

Starburst is sold in 2-packs, and there are 3 colors of starbust. What is the expected number of 2-packs you will buy if your goal is to get all colors?

Let the colors be ABC. In the first pack you can get one color or two colors.

$\frac{1}{3}$: 1

$\frac{2}{3}$: 2

$$P[1] = \frac{1}{3}$$
$$P[2] = \frac{2}{3}$$

$$E[x] = E[x|1]\,P[1] + E[x|2]\,P[2]$$
$$= E[x|1]\cdot\frac{1}{3} + E[x|2]\frac{2}{3}.$$

$E[x|2] \rightarrow$ waiting for one color. Success probability $= 1-\left(\frac{2}{3}\right)^2 = \frac{5}{9}$

$$\therefore\ E[x|2] = 1 + \frac{9}{5}$$

Compute $E[x|1]$: let us say the color you get is A. $E[x|1] = 1 + x$.

Compute x:

A → 0 new colors / 1 new colors / 2 new colors

$$x = (1+\text{Wait})\cdot P[0] + (1+\text{Wait})\cdot P(1) + (1+\text{Wait})P(2)$$
$$= (1+x)\,\underbrace{P(0)}_{\frac{1}{9}:\ AA} + (1 + \text{Wait to } 1)\,\underbrace{P(1)}_{\frac{6}{9}} + 1\cdot \underbrace{P(2)}_{\left.\begin{matrix}BC\\CB\end{matrix}\right\}\to\frac{2}{9}}$$
$$= \frac{1+x}{9} + (1 + \underbrace{\text{Wait to } 1}_{\frac{9}{5}\text{ (see above)}})\cdot\frac{6}{9} + 1\cdot\frac{2}{9}$$
$$= \frac{1+x}{9} + \left(1+\frac{9}{5}\right)\cdot\frac{6}{9} + \frac{2}{9}.$$
$$= \frac{1}{9} + \frac{6}{9} + \frac{2}{9} + \frac{x}{9} + \frac{6\cdot 9}{5\cdot 9}$$

$$\frac{8}{9}x = 1 + \frac{6\cdot 9}{5\cdot 9} = \frac{11}{5} \Rightarrow x = \frac{9}{8}\left(1+\frac{6\cdot 9}{5\cdot 9}\right) = \left(9 + \frac{6\cdot 9}{5}\right)\cdot\frac{1}{8} = \frac{99}{40}.$$

$$\therefore E[x|1] = \frac{139}{40}$$

$$\therefore E[x] = \frac{14}{5}\cdot\frac{2}{3} + \frac{139}{40}\cdot\frac{1}{3} = \frac{28}{15} + \frac{139}{120} = \frac{363}{120} = \frac{121}{40} = \boxed{3.025}$$

50% : Show Understanding

80% : MADE SIGNIFICANT PROGRESS

100% : ★CORRECT★ SOLUTION Typos OK.

5 DFA or no DFA

Give a DFA for $\mathcal{L} = \{0^{\bullet n^2} | n \geq 1\} = \{0, 0000, 000000000, \ldots\}$, or prove that $\mathcal{L}$ can't be solved with DFA.

Can't Be done.

Proof By contradiction suppose a DFA with K states solves $\mathcal{L}$.

Consider

state(0^1) state(0^2) st(0^3) .... state(0^{k+1}) ← By pigeonhole two states match.

state(0^i) = state(0^j) = q $i < j$

So from q, j-i 0's take you back to q.

That means $q = \text{state}(0^{i+m(j-i)})$ and $1 \leq j-i \leq K$.

$\text{state}(0^{i+n^2-i})$ = state(0^{i^2}) must be a yes state for $n \geq i$

$\therefore$ $\text{state}(0^{\underbrace{i+m(j-i)}_{q} + \underbrace{n^2-i}_{\text{process same \# of 0's}}})$ must be a yes state

$\therefore$ $n^2 + m(j-i)$ is a square for all $m \geq 0$

Choose $n = K$. Then

K^2 and $K^2 + (j-i)$ are both squares

but the next square after K^2 is $(K+1)^2 = K^2 + 2K + 1$.

this means $j-i \geq 2K+1$
but $j-i \leq K$ } FISHY ∎

50% Tinkering to show understanding or attempt to construct.
80% Reasonable attempt to prove
100% Correct proof

6 Transducer Turing Machine for Reversal.

Give a high level pseudo-code description of a transducer Turing Machine for reversal. The input on the tape is any binary string w. When the Turing Machine halts, the reversal of w should have replaced w. E.g:

Start

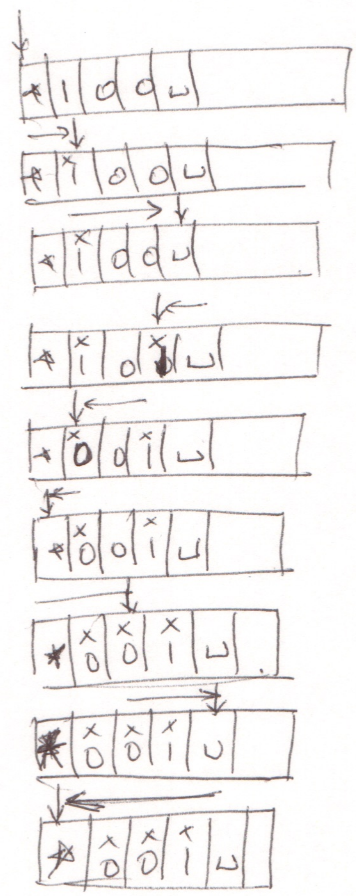
*	1	0	1	0	0	1	1	⊔

End

*	1	1	0	0	1	0	1	⊔

(Don't give machine level details, but you should make it clear how the Turing Machine moves back and forth. Tinker.)

1. From *
2. Move right to first unmarked bit
   - Come to ⊔ → done
   - Come to unmarked bit
     → remember bit
     → mark it
     → move right to ⊔
3. Move left to first unmarked bit
   - Come to * → done
   - Come to unmarked bit
     → remember bit
     → mark and replace with previously remembered bit in step 2
4. Move left to first marked bit
   - replace with remembered bit in step 3
   - Go back to *



DONE

50% tinkering with pictures

80% Reasonable Idea.

100% Reasonable solution