Final Exam

170 Minutes

First Name: Solutions

Last Name: ________

RIN: ________

NO COLLABORATION or electronic devices.
Any violations will result in an F.
No questions allowed during the test unless you think there is a mistake.

GOOD LUCK!

10 points per correct multiple-choice answer. Circle exactly one answer.
20 points per correct answer to Problems 2-6.
You **MUST** show **CORRECT** work to get credit.
Correct answers with no explanation will get a 0.

1	2	3	4	5	6	Total
200	20	20	20	20	20	300

SHOW WORK

1. What is the asymptotic behavior of the sum $S(n) = \sum_{i=1}^{n} i^9 + 100i^8$?

A $O(n^9)$
B $O(n^8)$
C $O(100n^9)$
D $O(n^{10})$
E None of the above.

$$\sum_{i=1}^{n} i^9 + 100 i^8 \approx \int_1^n x^9 + 100x^8\,dx = \left[\frac{x^{10}}{10} + \frac{100x^9}{9}\right]_1^n$$

$$= O(n^{10})$$

2. What can we say about this statement: $\exists C > 0 : \forall n \geq 1 : n^2 < \frac{1}{2}Cn^2 + n$?

A True
B False
C Depends on C
D Depends on n
E None of the above.

Pick any $C > 2$

3. What can we say about this statement: $\exists C > 0 : \forall n \geq 1 : n^3 < \frac{1}{2}Cn^2 + n$?

A True
B False
C Depends on C
D Depends on n
E None of the above.

n^3 eventually dominates n^2 for any C.

SHOW WORK

4. Consider the recurrence $T_1 = 1, T_n = T_{n-1} + n^3$. Estimate T_{10}:

A 250
B 2500
C 25000
D 250000
E None of the above.

$$T_n = T_{n-1} + n^3$$
$$T_{n-1} = T_{n-2} + (n-1)^3$$
$$\vdots$$
$$T_1 = 1.$$

$$\Rightarrow T_n = \sum_{i=1}^{n} i^3 \approx \int_1^n x^3\,dx.$$

$$\Downarrow$$

$$\int_1^{10} x^3\,dx = \left.\frac{x^4}{4}\right|_1^{10} = \frac{10000-1}{4}$$

$$\approx 2500$$

SHOW WORK

5. Calculate the sum $\sum_{i=1}^{5}\sum_{j=0}^{3} i2^j$.

A 200
B 225
C 250
D 275
E None of the above.

$$\sum_{i=1}^{5}\sum_{j=0}^{3} i2^j = \sum_{i=1}^{5} i \sum_{j=0}^{3} 2^j = \sum_{i=1}^{5} i\,(2^4-1)$$

$$= 15 \cdot \frac{5 \cdot 6}{2} = 225$$

Tinker: $3^2 = 9$

$3^3 = 27 \equiv 7 \pmod{10}$

$\Rightarrow \quad 3^4 = 81 \equiv 1 \pmod{10}$

$3^{100} = (3^4)^{25} \equiv 1^{25} \pmod{10}$

SHOW WORK

6. What is the last digit of 3^{100}?

   - A 1
   - B 3
   - C 5
   - D 7
   - E 20

7. Consider a graph G where every vertex has degree 2. What do we know?

   - A G must have at least 3 vertices. ✓ (2 vertices can't have degree
   - B G can be a cycle (i.e., the entire graph can be a single cycle). ✓

   

   - C G must have a cycle. ✓ (hw problem)
   - D G is not a tree. ✓ (a leaf has degree 1)
   - E All of the above.

8. Consider a graph G where every vertex has degree 3. What do we know?

   - A G must have an even number of vertices. handshaking thm
   - B G can be a tree.
   - C G cannot have any cycles.
   - D G cannot exist.
   - E All of the above.

SHOW WORK

9. How many graphs with 5 vertices are there? In total # edges $= \binom{5}{2} = 10$

   Each edge can either be added or removed so # graphs $= 2^{10} = 1024$

   - A 32
   - B 128
   - C 256
   - D 1024
   - E None of the above.

SHOW WORK

10. Suppose FOCS has 300 students, and I split them in 3 sections with 100 students each. How many ways to split the students are there?

    Annagrams. All words w/ 100 1s, 2s, 3s

    $\binom{300}{100,100,100}$

    - A $\binom{300}{100}$
    - B $\binom{300}{200}$
    - C $\binom{300}{100,100,100}$
    - D 300!
    - E None of the above.

SHOW WORK

11. You flipped 4 fair coins. What is the probability of exactly 3 heads?

- [A] 1/16
- [B] 2/16
- [C] 3/16
- [D] 4/16
- [E] None of the above.

$$\binom{4}{3} = \frac{4!}{3!\,1!} = 4$$

SHOW WORK

12. You flipped 4 fair coins. What is the probability of exactly 3 heads, given that the first coin was H?

- [A] 1/8
- [B] 2/8
- [C] 3/8
- [D] 4/8
- [E] None of the above.

$$\mathbb{P}[3H \mid c_1 = H] = \frac{\mathbb{P}[3H \wedge c_1 = H]}{\mathbb{P}[c_1 = H]} = \frac{\frac{3}{16}}{\frac{1}{2}} = \frac{3}{8}$$

2 more Hs in 3 flips, so 3 outcomes

13. Suppose $\mathbb{E}[X] = 3, \mathbb{E}[Y] = 2$. What is $\mathbb{E}[2X + 5Y]$?

- [A] 10
- [B] 16
- [C] 22
- [D] 28
- [E] None of the above.

$$\mathbb{E}[2X + 5Y] = 2\,\mathbb{E}[X] + 5\,\mathbb{E}[Y] = 16$$

SHOW WORK

14. Let X be a positive random variable. If $\sigma^2(X) = 3$ and $\mathbb{E}[X^2] = 5$, what is $\mathbb{E}[X]$?

- [A] $\sqrt{2}$
- [B] 2
- [C] 3
- [D] 4
- [E] None of the above.

$$\sigma^2(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$(\mathbb{E}[X])^2 = 5 - 3 = 2$$

$$\mathbb{E}[X] = \sqrt{2}$$

SHOW WORK

15. On any day, with probability 0.5 I go to the cafeteria at noon: if I go at noon, I get my food at 12:10pm with probability 0.2 and at 12:30pm with probability 0.8; with probability 0.5, I go to the cafeteria at 12:30pm and get my food at 12:30pm. How many days am I expected to wait until I get my food at 12:10pm (start counting from 1)?

- [A] 5
- [B] 10
- [C] 15
- [D] 20
- [E] None of the above.

Success prob.: $0.5 \times 0.2 = 0.1 = p$

go at noon   get food by 12:10pm

$$\mathbb{E}[\text{waiting time}] = \frac{1}{p} = 10$$

16. What do we know about the language $\mathcal{L} = \{w\#w \mid w \in \{0,1\}^*\}$?

- $\boxed{A}$ It is regular. Needs memory
- $\boxed{B}$ It is context-free. Can't be done w/ a stack
- $\boxed{C}$ It is decidable. showed TM in class
- $\boxed{D}$ It is undecidable.
- $\boxed{E}$ None of the above.

17. What do we know about the language $\mathcal{L} = \{w\#w^R \mid w \in \{0,1\}^*\}$?

- $\boxed{A}$ It is regular. Need memory
- $\boxed{B}$ It is context-free. showed PDA in class
- $\boxed{C}$ It is decidable.
- $\boxed{D}$ It is undecidable. 5 pts if you answered this since language is also decidable
- $\boxed{E}$ None of the above.

18. Can a pushdown automaton (PDA) solve the language $\mathcal{L} = \{ww^R \mid w \in \{0,1\}^*\}$?

- $\boxed{A}$ No, the language is not context-free.
- $\boxed{B}$ Yes, a deterministic PDA can solve $\mathcal{L}$. Needs to guess when w^R starts
- $\boxed{C}$ Yes, a non-deterministic PDA can solve $\mathcal{L}$.
- $\boxed{D}$ No, the language is not decidable.
- $\boxed{E}$ None of the above.

19. Which of the following problems is decidable?

- $\boxed{A}$ The halting problem. X
- $\boxed{B}$ Deciding whether a given program will print "Hello World". X (auto-grade)
- $\boxed{C}$ Deciding whether a given program will terminate. X (halting problem)
- $\boxed{D}$ Deciding whether a given C program is well-formatted. CFG
- $\boxed{E}$ None of the above.

20. What do we know about the traveling salesman problem?

- $\boxed{A}$ It is known to be in the class P. X (It's NP-complete, not known to be in P)
- $\boxed{B}$ I can solve it by listing all possible trajectories.
- $\boxed{C}$ It is undecidable.
- $\boxed{D}$ I can always list all trajectories in linear time. ← exponentially many
- $\boxed{E}$ None of the above.

Problem 2. Prove that $\forall n \geq 1, \log_2(n) \leq n$.

Using induction.

Base case: $n=1$.

$\log_2(1) = 0 \leq 1$. V.

50 % progress

Induction step:

Assume $P(n)$: $\log_2(n) \leq n$.

Need to prove $P(n+1)$: $\log_2(n+1) \leq n+1$.

80 % Understood problem. Significant progress

$\log_2(n) \leq n$

$2^{\log_2(n)} \leq 2^n$

$n \leq 2^n$

$n+1 \leq 2^n + 1$

$n+1 \leq 2^n + 2^n$

$n+1 \leq 2^{n+1}$

$\log_2(n+1) \leq \log_2(2^{n+1})$

$\log_2(n+1) \leq n+1$. V.

100 % Correct proof.

Problem 3. Suppose you are determined to flip a fair coin until you get the sequence HHH. What is the expected number of flips?

50 % progress

$$\mathbb{E}[3H] = (\underbrace{\mathbb{E}[3H \mid T]}_{\mathbb{E}[3H]} + 1) \cdot \mathbb{P}[T] + (1 + \underbrace{\mathbb{E}[3H \mid H]}_{\mathbb{E}[2H]}) \cdot \mathbb{P}[H]$$

80 % Significant progress

$$\mathbb{E}[2H] = (1 + \underbrace{\mathbb{E}[2H \mid T]}_{\mathbb{E}[3H]}) \cdot \mathbb{P}[T] + (1 + \underbrace{\mathbb{E}[2H \mid H]}_{\mathbb{E}[1H]}) \cdot \mathbb{P}[H]$$

~~$\mathbb{E}[1H] = 2$. (probability of succ~~

$$\mathbb{E}[1H] = (1 + \underbrace{\mathbb{E}[1H \mid T]}_{\mathbb{E}[3H]}) \cdot \mathbb{P}[T] + \cancel{\mathbb{E}} \, 1 \cdot \mathbb{P}[H]$$

100% correct derivation

Three equations and three unknowns.

Call $\mathbb{E}[3H] = x$.

$$\mathbb{E}[1H] = (1 + x) \cdot \frac{1}{2} + \frac{1}{2} = 1 + \frac{x}{2}$$

$$\mathbb{E}[2H] = (1 + x)\frac{1}{2} + \left(1 + 1 + \frac{x}{2}\right) \cdot \frac{1}{2} = \frac{3}{2} + \frac{3}{4}x$$

$$\mathbb{E}[3H] = x = (1 + x) \cdot \frac{1}{2} + \left(1 + \frac{3}{2} + \frac{3}{4}x\right) \cdot \frac{1}{2}$$

$$\frac{x}{8} = \frac{1}{2} + \frac{5}{4} = \frac{14}{8}$$

$$\boxed{x = 14}$$

Problem 4. Prove that for any $n \geq 3$, there exists a set of n distinct natural numbers $x_1, \dots, x_n$ such that each x_i divides the sum $s = x_1 + \cdots + x_n$, i.e., $s = x_i k_i$ for some $k_i \in \mathbb{N}$. Tinker, tinker, tinker.

Tinker.

$n=3$: $1+2+3=6$

divide 6

$n=4$: $1+2+3+6=12$

divide 12.

Guess: $x_1 = 1, x_2 = 2, x_3 = 3$.

$x_n = x_{n-1} + \cdots + x_1$.

Proof using induction.

Base case: $n=3$. V.

Induction step:

Assume $P(n)$: $x_1 + \cdots + x_n = s_n$

& $x_i \mid s_n$.

Prove $P(n+1)$. $[x_{n+1} = x_n + \cdots + x_1 = s_n]$.

We know $x_1 + \cdots + x_n + x_{n+1} = 2x_{n+1}. = s_{n+1}$

So $x_{n+1} \mid s_{n+1}$.

Also $s_{n+1} = 2s_n$.

From induction hypothesis, all $x_i \mid s_n$, $i \leq n$.

50 %
Identify some cases

80 %
Correct guess and proof setup

100 %
Correct proof.

Problem 5. Consider the language of all odd-length zero strings $\mathcal{L}_O = \{0, 000, 00000, \dots\}$. Prove that $\mathcal{L}_O$ has an undecidable subset.

First note $\mathcal{L}_0$ is countable.

$w_i = 0 \cdot (2i-1)$

List $\mathcal{L}_0 = \{w_1, w_2, w_3, \dots\}$.

50 % Progress

Consider any subset $S \subseteq \mathcal{L}_0$:

$\mathcal{L}_0$	w_1	w_2	w_3	w_4	...
S	0	1	1	0	...

Suppose $S = \{w_2, w_3, \dots\}$.
I can represent S as an infinite binary string.

80 % Recognize the correct mapping.

The set of all subsets of $\mathcal{L}_0$ can be mapped, with a bijection, to the set of all infinite binary strings.

uncountable.

100 % Correct conclusion

TMs are countable, so some language must be undecidab

Problem 6. Consider the language $\mathcal{L}_{\text{AddTwo}} = \{0^{\bullet n}1^{\bullet n+2}\}$. Give pseudocode for a Turing Machine that decides this language.

[Your pseudocode needs to be detailed enough so it is clear that each step can indeed be performed using a Turing Machine.]

Step 1. Check if input has correct format.
(0s ~~folled~~ followed by 1s).
A DFA can do this.
If wrong format, REJECT.
If correct format, return to *.
Go to step 2.

50 %
Progress.

Step 2. Go right to the first unmarked 0.
Mark 0. Proceed to step 3.
If no unmarked 0's, proceed to step ~~1~~ 4.

Step 3. Go right to first unmarked 1.
If found an unmarked 1, mark it and return to * and step 2.
If no unmarked 1s, REJECT.

80 %
Good progress

Step 4. Go right to ~~last~~ first unmarked 1. Check the remaining string has exactly 2 1's. A DFA can ~~do~~ do this. If yes, ACCEPT, otherwise REJECT.

100 %.
All steps present & clear.

Scratch

Scratch