Final Exam

170 Minutes

First Name: ______________________

Last Name: ______________________

RIN: ______________________

NO COLLABORATION or electronic devices.
Any violations will result in an F.
No questions allowed during the test unless you think there is a mistake.

GOOD LUCK!

10 points per correct multiple-choice answer. Circle exactly one answer.
20 points per correct answer to Problems 2-6.
You **MUST** show **CORRECT** work to get credit.
Correct answers with no explanation will get a 0.

1	2	3	4	5	6	Total
200	20	20	20	20	20	300

1. What is the asymptotic behavior of the sum $S(n) = \sum_{i=1}^{n} i^9 + 100i^8$?

   - A $O(n^9)$
   - B $O(n^8)$
   - C $O(100n^9)$
   - D $O(n^{10})$
   - E None of the above.

2. What can we say about this statement: $\exists C > 0 : \forall n \geq 1 : n^2 < \frac{1}{2}Cn^2 + n$?

   - A True
   - B False
   - C Depends on C
   - D Depends on n
   - E None of the above.

3. What can we say about this statement: $\exists C > 0 : \forall n \geq 1 : n^3 < \frac{1}{2}Cn^2 + n$?

   - A True
   - B False
   - C Depends on C
   - D Depends on n
   - E None of the above.

4. Consider the recurrence $T_1 = 1, T_n = T_{n-1} + n^3$. Estimate T_{10}:

   - A 250
   - B 2500
   - C 25000
   - D 250000
   - E None of the above.

5. Calculate the sum $\sum_{i=1}^{5} \sum_{j=0}^{3} i2^j$.

   - A 200
   - B 225
   - C 250
   - D 275
   - E None of the above.

6. What is the last digit of 3^{100}?

- A 1
- B 3
- C 5
- D 7
- E 20

7. Consider a graph G where every vertex has degree 2. What do we know?

- A G must have at least 3 vertices.
- B G can be a cycle (i.e., the entire graph can be a single cycle).
- C G must have a cycle.
- D G is not a tree.
- E All of the above.

8. Consider a graph G where every vertex has degree 3. What do we know?

- A G must have an even number of vertices.
- B G can be a tree.
- C G cannot have any cycles.
- D G cannot exist.
- E All of the above.

9. How many graphs with 5 vertices are there?

- A 32
- B 128
- C 256
- D 1024
- E None of the above.

10. Suppose FOCS has 300 students, and I split them in 3 sections with 100 students each. How many ways to split the students are there?

- A $\binom{300}{100}$
- B $\binom{300}{200}$
- C $\binom{300}{100,100,100}$
- D 300!
- E None of the above.

11. You flipped 4 fair coins. What is the probability of exactly 3 heads?

   A 1/16
   B 2/16
   C 3/16
   D 4/16
   E None of the above.

12. You flipped 4 fair coins. What is the probability of exactly 3 heads, given that the first coin was H?

   A 1/8
   B 2/8
   C 3/8
   D 4/8
   E None of the above.

13. Suppose $\mathbb{E}[X] = 3, \mathbb{E}[Y] = 2$. What is $\mathbb{E}[2X + 5Y]$?

   A 10
   B 16
   C 22
   D 28
   E None of the above.

14. Let X be a positive random variable. If $\sigma^2(X) = 3$ and $\mathbb{E}[X^2] = 5$, what is $\mathbb{E}[X]$?

   A $\sqrt{2}$
   B 2
   C 3
   D 4
   E None of the above.

15. On any day, with probability 0.5 I go to the cafeteria at noon: if I go at noon, I get my food at 12:10pm with probability 0.2 and at 12:30pm with probability 0.8; with probability 0.5, I go to the cafeteria at 12:30pm and get my food at 12:30pm. How many days am I expected to wait until I get my food at 12:10pm (start counting from 1)?

   A 5
   B 10
   C 15
   D 20
   E None of the above.

16. What do we know about the language $\mathcal{L} = \{w\#w \mid w \in \{0,1\}^*\}$?

   A It is regular.
   B It is context-free.
   C It is decidable.
   D It is undecidable.
   E None of the above.

17. What do we know about the language $\mathcal{L} = \{w\#w^R \mid w \in \{0,1\}^*\}$?

   A It is regular.
   B It is context-free.
   C It is decidable.
   D It is undecidable.
   E None of the above.

18. Can a pushdown automaton (PDA) solve the language $\mathcal{L} = \{ww^R \mid w \in \{0,1\}^*\}$?

   A No, the language is not context-free.
   B Yes, a deterministic PDA can solve $\mathcal{L}$.
   C Yes, a non-deterministic PDA can solve $\mathcal{L}$.
   D No, the language is not decidable.
   E None of the above.

19. Which of the following problems is decidable?

   A The halting problem.
   B Deciding whether a given program will print "Hello World".
   C Deciding whether a given program will terminate.
   D Deciding whether a given C program is well-formatted.
   E None of the above.

20. What do we know about the traveling salesman problem?

   A It is known to be in the class P.
   B I can solve it by listing all possible trajectories.
   C It is undecidable.
   D I can always list all trajectories in linear time.
   E None of the above.

Problem 2. Prove that $\forall n \geq 1, \log_2(n) \leq n$**.**

Problem 3. **Suppose you are determined to flip a fair coin until you get the sequence HHH. What is the expected number of flips?**

Problem 4. Prove that for any $n \geq 3$, there exists a set of n distinct natural numbers $x_1, \ldots, x_n$ such that each x_i divides the sum $s = x_1 + \cdots + x_n$, i.e., $s = x_i k_i$ for some $k_i \in \mathbb{N}$. Tinker, tinker, tinker.

Problem 5. Consider the language of all odd-length zero strings $\mathcal{L}_O = \{0, 000, 00000, \dots\}$. Prove that $\mathcal{L}_O$ has an undecidable subset.

Problem 6. Consider the language $\mathcal{L}_{\textbf{AddTwo}} = \{0^{\bullet n}1^{\bullet n+2}\}$. Give pseudocode for a Turing Machine that decides this language.

[Your pseudocode needs to be detailed enough so it is clear that each step can indeed be performed using a Turing Machine.]

Scratch

Scratch