1 Digraphs

Definition 1

A **digraph** or **directed graph** G is a triple comprised of a **vertex set** $V(G)$, **edge set** $E(G)$, and a function assigning each edge an ordered pair of vertices (**tail**, **head**); these vertices together are called **endpoints** of the edge. We say that an edge is from its tail to its head.

In a digraph, an edge is a loop if its endpoints are equal. Multiple edges are those with identical tails and identical heads.

A digraph is called simple, if it has no multiple edges. An edge with the tail u and the head v is denoted uv. Vertex u is called the **predecessor** of v, and v is called the **successor** of u.

Definition 2

The **underlying graph** of a digraph D is the graph G obtained from D by considering edges of D as unordered pairs.

A digraph is **weakly connected** if its underlying graph is connected.

A digraph is **strongly connected**, or **strong**, if for every **ordered pair** of **distinct** vertices x and y, there is a directed path starting at x and ending at y.

The **strong components** of a digraph are its **maximal** strong subgraphs.

A digraph without cycles is called a **DAG** (directed acyclic graph).

Theorem 1 *Let S_1 and S_2 be two strong components of a digraph G. Then $S_1 \cap S_2 = \emptyset$.*

Prove it. ∎

Theorem 2 *Let $S_1, S_2, \ldots, S_k$ be the set of all strong components of a digraph G. Form a new digraph H whose vertices are $S_1, \ldots, S_k$, and the edges are the set of all ordered pairs (S_i, S_j) such that, in graph G, there is an edge xy where $x \in S_i$ and $y \in S_j$. Then H is a DAG.*

Prove it. ∎

Definition 3

Let D be a digraph and $u, v \in V(D)$. The **outdegree** $d^+(u)$ (resp. **indegree** $d^-(v)$) of u is the number of edges with u as the tail (resp. v as the head).

Problem 1 *In a digraph D,*

$$\sum_{v \in V(D)} d^+(v) = |E(D)| = \sum_{u \in V(D)} d^-(u)$$

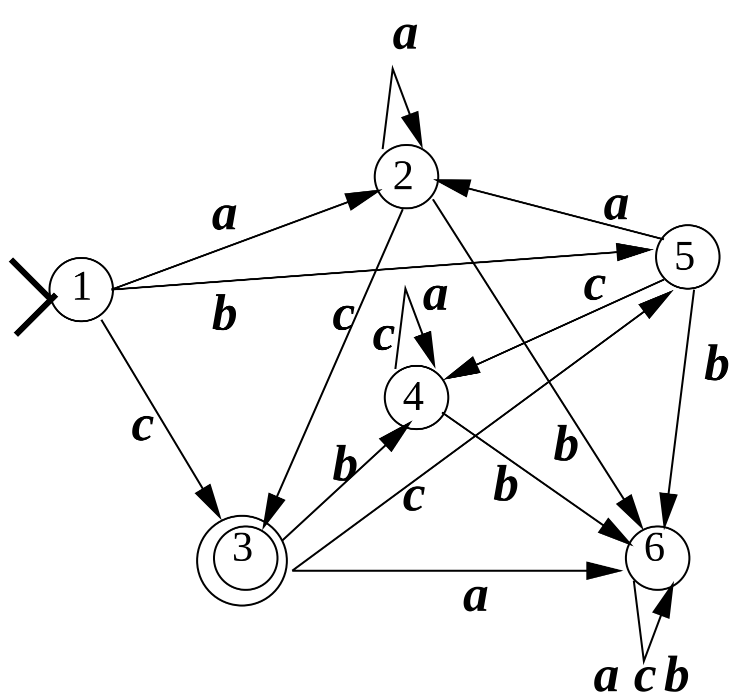
Prove it. ∎

Definition 4

A digraph is called a **tournament** iff for every two vertices, the digraph contains exactly one directed edge.

Definition 5

Let G be a digraph of the outdegree k for some integer $k > 0$ whose edges are labeled with symbols $a_1, a_2, \ldots, a_k$ so that for every vertex $x \in V(G)$, the labels on the edges leaving x are $a_1, a_2, \ldots, a_k$. Such a graph is called an **automaton** graph.



Subgraphs; Isomorphism; decomposition; union the same for graphs and digraphs.

Two digraphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are **isomorphic** if there is a one-to-one mapping $f : V_1 \to V_2$ such that for all u and $v \in V_1$, $(uv) \in E_1$ iff $(f(u)f(v)) \in E_2$.

The **adjacency** matrix of a graph G: $A = (a_{i,j})$, where

$$a_{i,j} = \begin{cases} 1, & \text{iff } ij \in E \\ 0, & \text{else.} \end{cases}$$

In the case of a digraph with multiple edges, $a_{i,j}$ is the number of edges with tail i and head j. Unless it is specified, our digraphs have no multiple edges nor loops.

The **incidence matrix** $M(G) = (m_{i,j})$ of a graph G with vertices $v_1, \ldots, v_n$ and edges $e_1, \ldots, e_m$, is defined as follows

$$m_{i,j} = \begin{cases} +1, & \text{if } v_i \text{ is the tail of } e_j, \\ -1, & \text{if } v_i \text{ is the head of } e_j, \\ 0, & \text{else.} \end{cases}$$

In an **undirected** graph G, a **walk** is a list

$$v_1 e_1 v_2 e_2 \dots v_{k-1} e_k v_k$$

such that $v_1, \dots, v_k$ are vertices; $e_1, e_2, \dots, e_k$ are edges; and

$$\forall i = 1, \dots, k-1, e_i = v_i v_{i+1}.$$

The length of a walk is the number of edges.

A **trail** is a walk without repeated edges.

A **path** is a trail without repeated vertices.

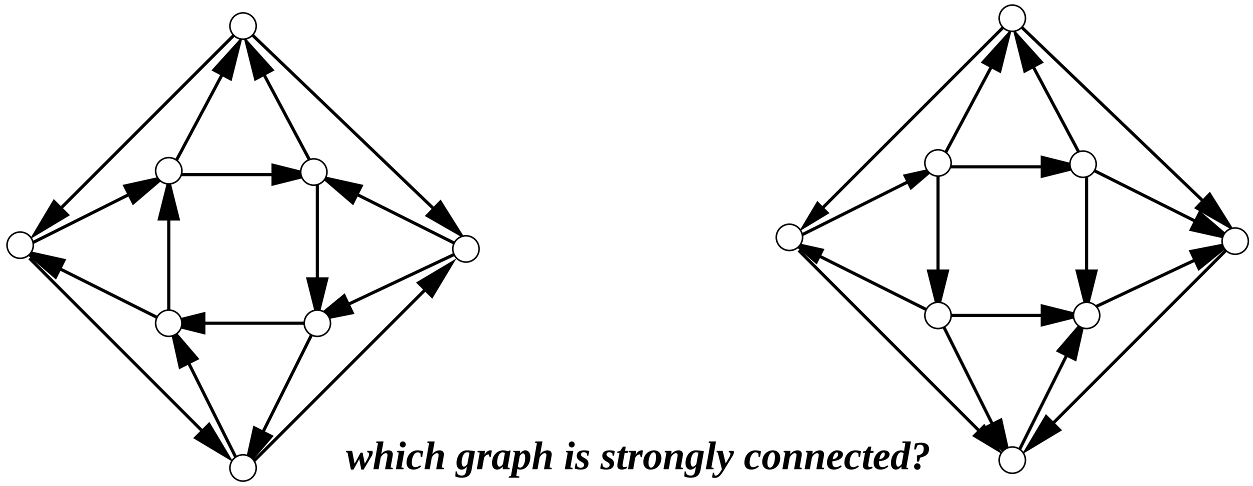
A trail whose first and last vertices are the same is called a **closed walk**, or a **circuit**.
A circuit without repeated vertices is called a **cycle**.

In a **directed** graph **walks, trails,** and **paths** are defined similarly satisfying the **follow_the_arrows** rule: the head of an edge is the tail of the next edge in the sequence.

Given a digraph G, its **underlying graph** G^* is obtained by replacing all directed edges with corresponding undirected edges.

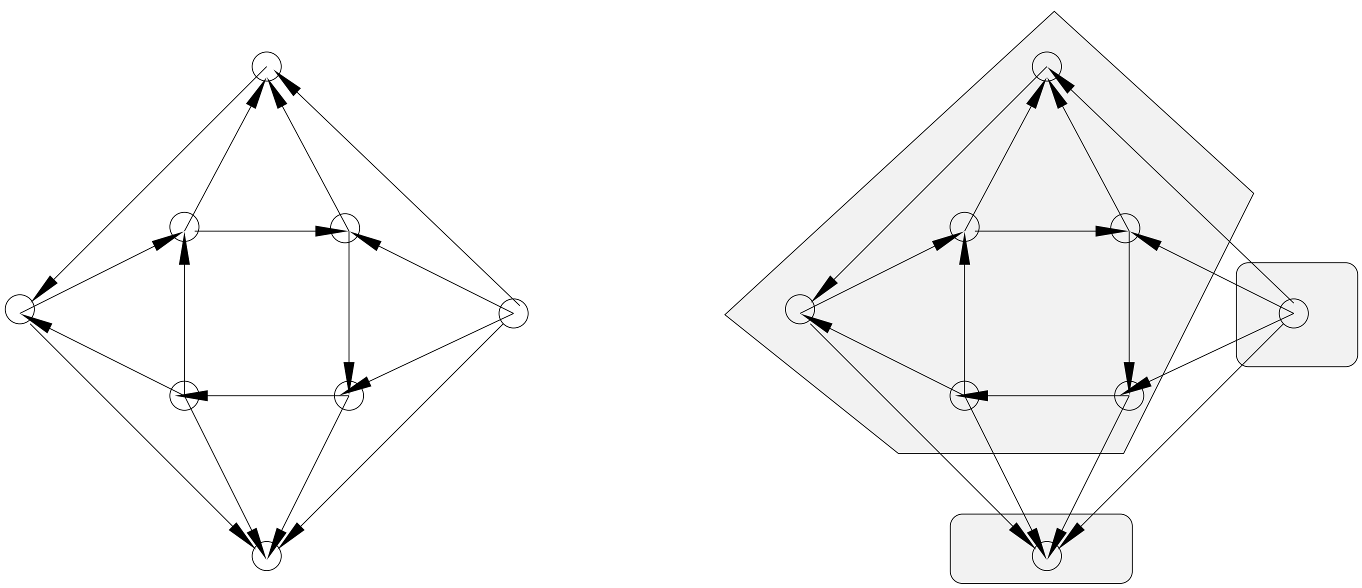
A **path** or **directed path** in a digraph G is a sequence of edges $\{e_i\}_{i=1}^{p}$ such that
for every $i = 1, \dots, p-1$, the head of e_i is the tail of e_{i+1}.

A digraph G is **strongly connected** or **strong** if for every x and y, there is a directed path starting at x and ending at y.



A subgraph H is a **strongly connected component** of a given digraph G if H is strong and no other strong subgraph contains H.

Theorem 3 *Every digraph G can be partitioned into strong connected components with disjoint sets of vertices.*



An **Eulerian trail** in a digraph is a trail which contains all edges.

Lemma 1 *Let G be a digraph with the smallest outdegree $d^+(G) \geq 1$. Then G has a cycle.*

Proof. Starting with an arbitrary vertex $x \in V(G)$, form a sequence of vertices as follows:

$$\begin{aligned} x_1 &= x; \\ x_{i+1} &= \text{the head of an edge } e_i \text{ whose tail is } x_i \end{aligned}$$

Terminate the sequence when the first member x_q is encountered which is equal to a member x_p already in the sequence.

Clearly the sequence $x_p e_p x_{p+1} \cdots e_{q-2} x_{q-1} e_{q-1} x_p$ is a cycle. ∎

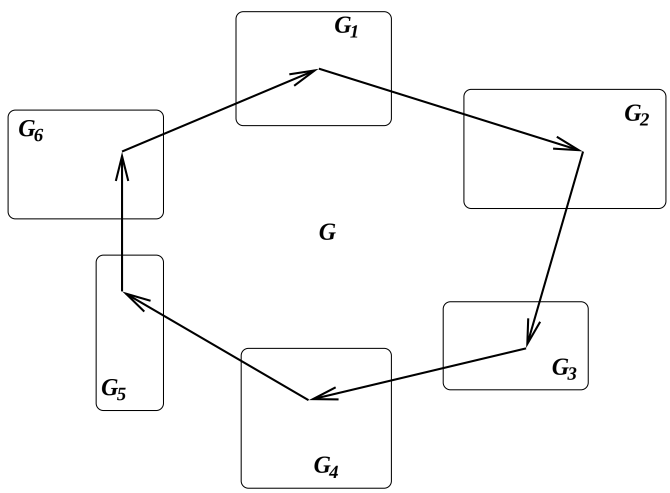
Theorem 4 *Let G be a digraph whose underlying graph is connected and has at least 2 vertices. Then G has an Eulerian circuit iff for every vertex $i \in [1, n]$, $d^+(i) = d^-(i)$.*

Proof. Induction on the number of edges of the digraph. Since the underlying graph is connected, for every vertex $v \in V(G)$, $d^+(v) \geq 1$. This implies that $|E(G)| \geq |V(G)|$.

Base. Suppose $|E(G)| = |V(G)|$. Then $d^+(v) = d^-(v) = 1$ for every vertex $v \in V(G)$. This implies that G is a directed cycle, which is also the Eulerian cycle of the graph.

Inductive step. Suppose the theorem holds for every graph with $< m$ edges, and let G be a graph with m edges which satisfies the condition of the Theorem.

Since $d^+(x) \geq 1$ for every $x \in V(G)$, by the Lemma 1 above, G has a directed cycle C. Consider now the graph $H = (V, E - E(C))$ which is obtained from G by removing all edges of C (see Figure below). Let $G_1, G_2, \ldots, G_k$ be the connected component of H. For every G_i ($i \in [1, k]$), which is not an isolated vertex, the conditions of the Theorem hold, and each of them has fewer than m edges (**explain the reasons**). Thus, by induction, each such component G_i has an Eulerian trail W_i. Then, combining all W_is with C yields an Eulerian trail for G. ▮



Theorem 5

A connected undirected graph is Eulerian if the degree of every vertex is even.

Fleury's Algorithm

Input: An undirected connected graph;

Output: An Eulerin trail, if it exists.

1. If there are vertices of odd degree, halt and return
   `The Graph is not Eulerian;`
2. Unmark all edges of G; choose any $v \in V(G)$; $i = 0$;
3. Select unmarked edge e incident to v which in not a *bridge* in the spanning subgraph comprised of unmarked edges; if such an edge does not exist, let e be any unmarked edge incident to v;
4. If $e = (v, u)$, then $C = Ceu$; $v = u$; $i = i + 1$; and mark e;
5. If $i = |E|$, then halt and output
   `C is an eulerian trail`;
   otherwise go to step 2;

2 Problems.

Proposition 1 *Let G be the graph with vertex set $\{1, 2, \ldots, 15\}$ in which integers i and j are adjacent iff they have a common factor exceeding 1. How many connected components does G have? What is the maximal length of a path in G?*

Proposition 2 *A digraph is called a* **tournament** *if its underlying graph is complete. A vertex v in a digraph is called a* **leader** *if every other vertex can be reached from v by a path of length at most 2. Prove that every tournament has a leader.*

Problem 2 *Prove that in every digraph, there is a strong component such that the digraph has no edges leaving the component; and there is a strong component such that the digraph has no edges entering the component.*

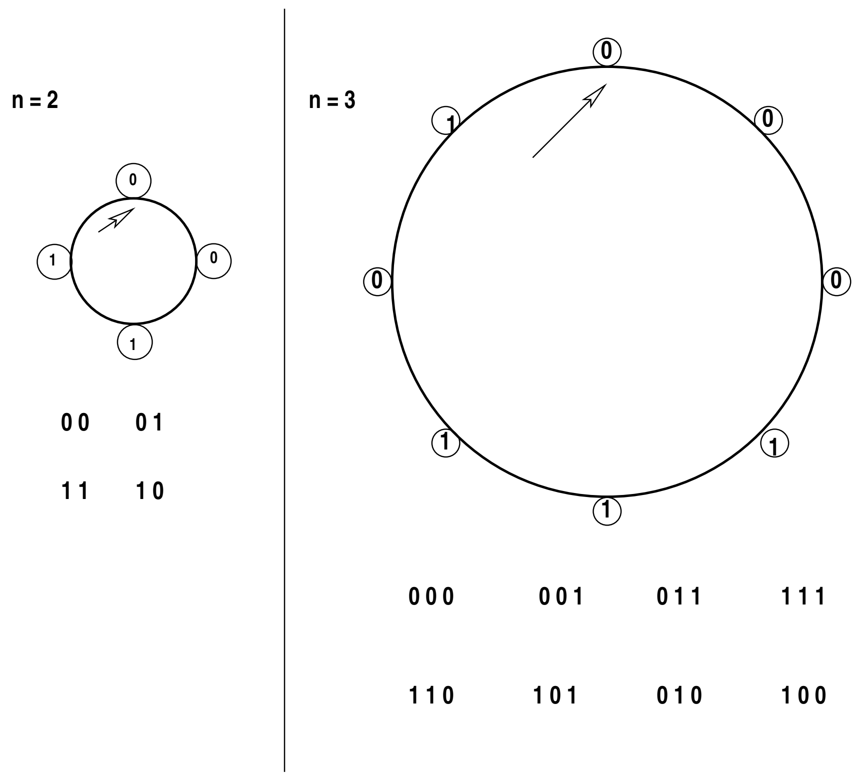
Problem 3 *Prove that in a digraph, every closed walk of an odd length contains a cycle of an odd length.*

Problem 4 *Let G be a digraph in which indegree equals outdegree at every vertex. Prove that G has a decomposition into cycles.*

Problem 5 *Prove that for every directed tree T on n vertices, there is a mapping $f : V(T) \to \{1, 2, \ldots, n\}$ such that for every directed edge $uv \in E(T)$, $f(u) < f(v)$.*

3 deBruin Graphs

Puzzle



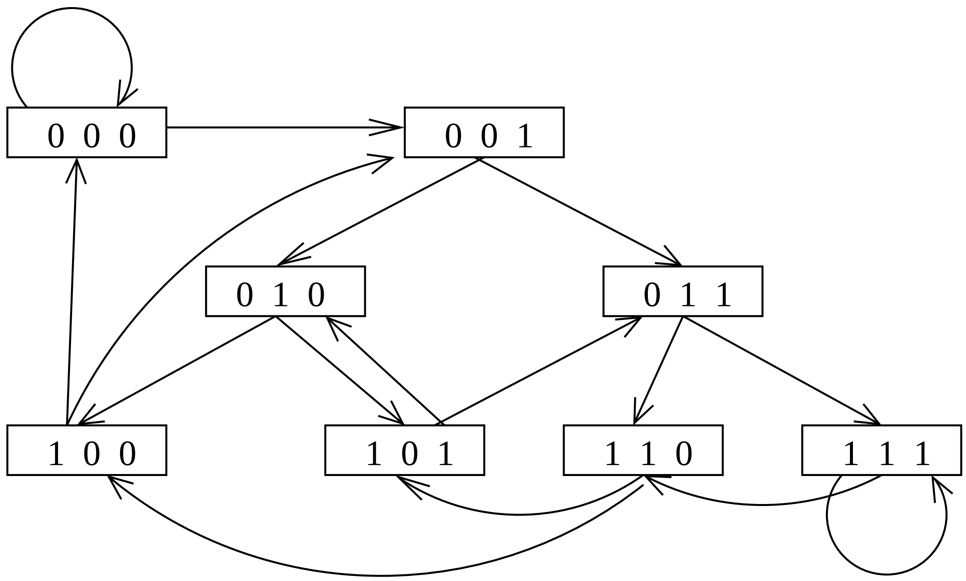
Can it be done for an arbitrary n?

We are looking for a 0,1-labelling of a cycle of length 2^n such that the 2^n intervals of length n of the cycle were all distinct?

deBruin graph D_n is a finite directed graph defined by

$V(D_n)$ the set of binary strings of length $n-1$;

$E(D_n) = \{(a_1, a_2, \ldots, a_{n-1}), (a_2, a_3, \ldots, a_n)$ where $a_i \in \{0, 1\}$.



Mark every edge by the string of length n whose first $n-1$ symbols are the label of the start vertex of the edge, and the last $n-1$ symbols is the label of the end vertex.

Then the task of the puzzle is to construct a closed trail in D_n which traverse every edge of D_n exactly once.

Questions:

1. what are the indegrees and outdegrees of D_n?

2. how many vertices does D_n have?

3. how many edges does D_n have?

4. what is the diameter of D_n?

The *diameter* of a directed graph is the largest distance between any two vertices, where the distance $dist(x, y)$ between x and y is defined as the length of a shortest directed path starting at x and ending at y.

The Main Question: how does the deBruin graph help to solve the puzzle?

Question 1: What is the necessary and sufficient condition for a graph to be Eulerian?

Question 2: Is there a fast algorithm to construct an eulerian trail if it exists?

4 Kautz Graphs

The Kautz digraph K_n is a directed graph, where

- $V(K_n)$ is the set of strings $\{x_1, \ldots, x_{n-1}\}$ of length $n-1$ over the alphabet $\{0, 1, 2\}$, such that $\forall i \in [1, n-2],\ x_i \neq x_{i+1}$; and

- the set of edges are pairs of strings $(x_1, \ldots, x_{n-1}; y_1, \ldots, y_{n-1})$, where

$$x_2 = y_1; \ldots ; x_{n-1} = y_{n-2}.$$