1 Non-deterministic Turing Machine

A **nondeterministic Turing machine** is a generalization of the standard TM for which every configuration may yield none, or one or **more than one** next configurations.

In contrast to the deterministic Turing machines, for which a computation is a sequence of configurations, a computation of a nondeterministic TM is a tree of configurations that can be reached from the start configuration.

In this tree, the children-nodes of a node are its next configurations. Thus, the configuration, whose state is either q_a, or q_r has no children-nodes.

A **nondeterministic Turing machine**, written NTM, is a 7-tuple

$$M = (Q, \Sigma, \Gamma, \Delta, q_0, q_a, q_r),$$

where all ingredients except for Δ are defined as before for the deterministic TM.

$$\Delta : (Q \times \Gamma) \longrightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\}).$$

The transition function of an NTM may correspond, for a given pair (q, σ), a set of triples $\{(p, \sigma', D)\}$; this set can be empty.

The transition function is sometimes presented as a set Δ of pairs:

$$((q, \sigma)(p, \sigma', D)).$$

While for a deterministic TM, there can be just one pair with a given first term (q, σ), for a non-deterministic TM, Δ may have more than one, or none, pair with a given first term.

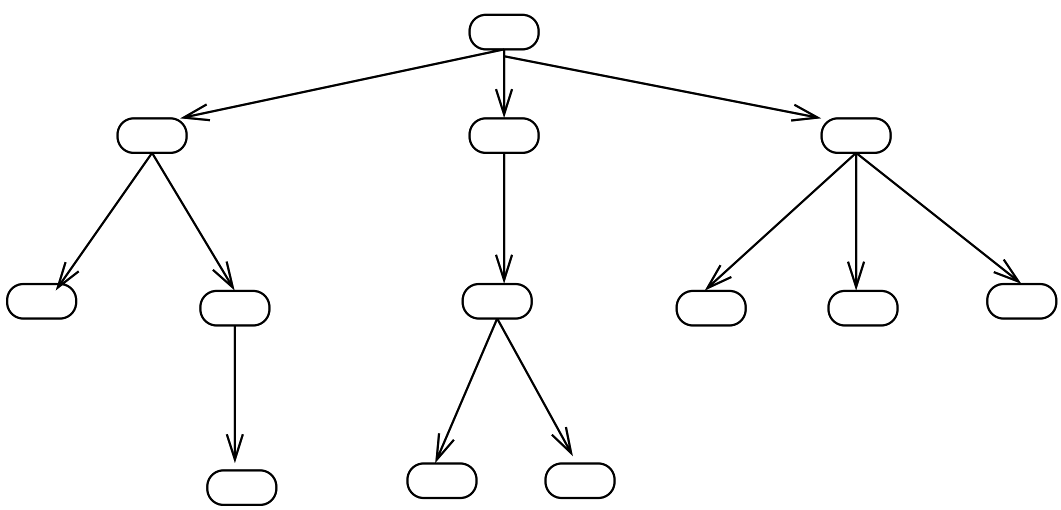
Thus, for each combination of a state and a tape-symbol, there can be more than one appropriate steps, or none at all. A configuration of an NTM may yield several, but a finite configurations in one step.

An input to an NTM is said to be **accepted** if **there exists at least one** node of the computation tree which is an accept-configuration.

The path from the root to the accept-configuration is said to be selected **non-deterministically**.

A non-deterministic Turing Machine is called a **decider** if all branches halt on all inputs.

If, for some input, **all branches** are rejected, then the input is rejected.



NDTM computation tree: schematic representation

Example. A high-level description of an NTM which accepts composite numbers L in the unary representation.

$$L = \{\underbrace{II \cdots I}_{m\ times} : \ m \text{ is a composite integer.}\}.$$

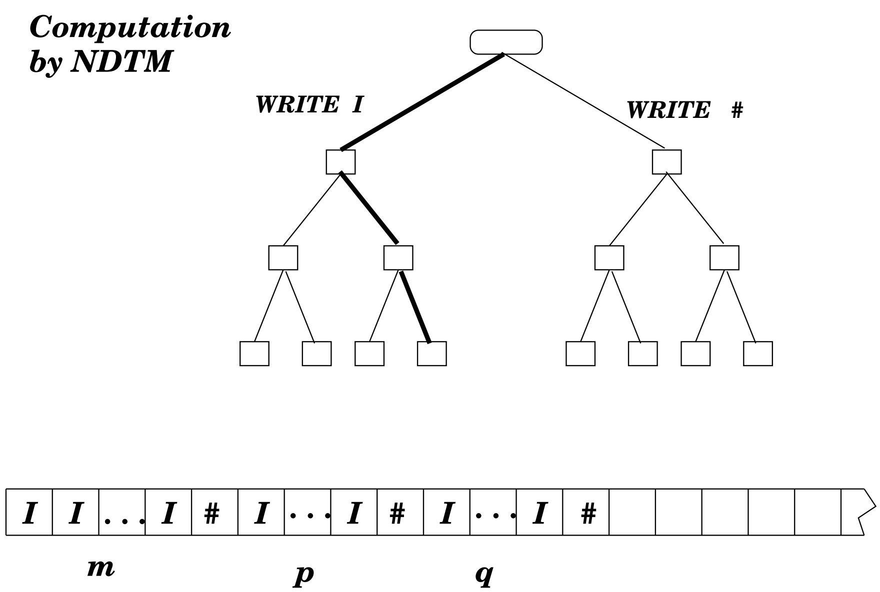
Given an input $\underbrace{II \ldots I}_{m\ times} \equiv I^m$, if m is not a prime, *i.e.* $m = p \times q$ for some integers $p, q < m$, the machine performs the following instructions:

1. Non-deterministically choose two integers p and q $(p, q \neq m)$;
2. transform the input configuration $(\epsilon\ q_{start}\ I^m)$ into $(I^m \# I^p \# I^q \#\ d\ \epsilon)$
   /* here # serves as a separator; d is a state of the control;*/
3. Multiply p and q, that is, transform $(I^m \# I^p \# I^q \#\ d\ \epsilon)$ into $(I^m \# I^{p \times q} \#\ f\ \epsilon)$;
4. Compare the length of m with $p \times q$: accept if they are the same, otherwise reject.

How to interpret the instruction "Non-deterministically choose two numbers p and q"?

To **non-deterministically** select p and q:

- starting from a given square, `repeat` p times: write I, move to the right.
- write #, move to the right.
- `repeat` q times: write I, move to the right.



1. Non-deterministically write in I^pI^q $(p > 1, q > 1)$
2. Deterministically generate in $I^{p \times q}$
3. Deterministically compare I^m with $I^{p \times q}$

Definition 1 *Two TM's M_1 and M_2 are said to be computationally equivalent, if $L(M_1) = L(M_2)$.*

Theorem 1

Every non-deterministic Turing machine has an equivalent deterministic Turing machine. If L is decided by an NDTM N in time $f(n)$, then there is a deterministic TM which decides L in time $\mathcal{O}(c^{f(n)})$, for some $c > 1$.

Proof. Let $N = (Q, \Sigma, \Gamma, \Delta, q_0, q_a, q_r)$ and let d be the smallest integer such that for each (q, σ) there is at most d choices for N.

A deterministic TM D traverses all nodes of the computational tree of N. Although the traversal can be exponentially longer than a path from the start configuration to the accepting configuration, it is *finite*, since every node of the tree has a **finite and bounded** number of children-nodes. The latter is determined by the cardinality of $|\delta(Q \times \Gamma)|$, which is a finite number since $\delta(Q \times \Gamma) \subseteq Q \times \Gamma \times \{L, R\}$).

D has three tapes (we already saw that such multi-tape TM is equivalent to a single tape TM):

input tape: contains the input w (never changes);

simulation tape: maintains a copy of N's tape on a branch of its nondeterministic computation;

address tape: keeps track of the current location of D in the N's computational tree; in particular, enumerates tree-paths in the lexicographical order.

D works as follows:

1. Insert the input w to tape 1; empty tapes 2 and 3.
2. Copy tape 1 to tape 2.
3. Simulate one branch of nondeterministic computation. Before each step of N, consult the next symbol on tape 3 to determine which choice to make among those allowed by N's transition function.
   - if no more symbols remain on tape 3 or if this nondeterministic choice is invalid, abort this branch by going to stage 4.
   - if a rejection configuration is encountered, go to stage 4.
   - if an accepting configuration is encountered, *accept* the input.
4. Replace the string on tape 3 with the lexicographically next string. Simulate the next branch of N's computation by going to stage 2.

Corollary. A language is Turing-recognizable (acceptable) iff some nondeterministic Turing machine recognizes (accepts) it.

2 Examples of non-deterministic Turing Machines

Example 1

Given a set $S = \{a_1, \dots, a_n\}$ of integers, determine if there is a subset $T \subseteq S$ such that

$$\sum_{a_i \in T} a_i = \sum_{a_i \in S-T} a_i.$$

The task is to construct an NDTM which accepts a **language** L corresponding to the problem.

Language:

$$L = \{a_1\#a_2\# \dots a_m\# : \exists T \subseteq S, \text{ such that } \sum_{a_i \in T} a_i = \sum_{a_i \in S-T} a_i.\}$$

Assume that there are two auxiliary TM (*deterministic*):

- C, a copy machine: copies a specified string on the tape to a specified location;
- Sum, a summation machine: sums up specified numbers.

NDTM which accepts L;

1. Place S onto the tape;
2. sum up the numbers in the input; let the result be A_1
3. append the string with \$;
4. while moving from the left to \$;
   nondeterministically copy all $a_i \in T$
   to the right from the rightmost #
   append with #
5. sum up the numbers that were copied; let the result be A_2;
6. accept if $A_1 = 2A_2$

Example 2

Given a graph $G = (V, E)$ and an integer $k > 0$, determine if there is a subset $C \subseteq V$ such that

- $|C| \geq k$;
- any two vertices in C are adjacent (C *is a clique*).

Present the problem as one of accepting a language L; describe an NDTM to accept L.

Solution. Define language L as follows:

$$L = \{\langle G, k\rangle : G \text{ has a clique of size} \geq k.\}$$

/* we assume that there is some standard way of presenting G as a string in a finite alphabet */

Assume there is a TM which, given two vertices of G, answers if these vertices are adjacent.

NDTM which accepts L;

1. Place $\langle G, k \rangle$ onto the tape;
2. append the string with \$;
3. while moving from the left to \$;

   nondeterministically select some vertices $v_i \in V(G)$

   /*assumption: there is a "finite-choice" computational
   path which does the selection, e.g.
   it can be just having a vector of length n with
   components 0 or 1 that define the selection */
4. check if the number of selected vertices $\geq k$;
5. for every two selected vertices check if they are adjacent;
6. accept if all pairs are adjacent

Example 3

Given a graph $G = (V, E)$ and an integer $k > 0$, determine if there is a path P in G such that

- the length of $P \geq k$;
- no two vertices in G are traced twice by P.

Present the problem as one of accepting a language L; describe an NDTM to accept L.

Solution. Define language L as follows:

$$L = \{\langle G, k\rangle : G \text{ there is a path of length} \geq k.\}$$

Assume there is a TM which, given two vertices of G, answers if these vertices are adjacent.

NDTM which accepts L;

1. Place $\langle G, k \rangle$ onto the tape;
2. append the string with \$;
3. while moving from the left to \$;
   nondeterministically select some vertices $v_i \in V(G)$
   /* we assume here without going into details that there is a "finite-choice" computational path which does the selection; for example, it can be just having a vector of length n with components 0 or 1 that define the selection */
4. check if the number of selected vertices $\geq k$;
5. for every two consequently selected vertices check if they are adjacent;
6. accept if all checked pairs of vertices are adjacent.