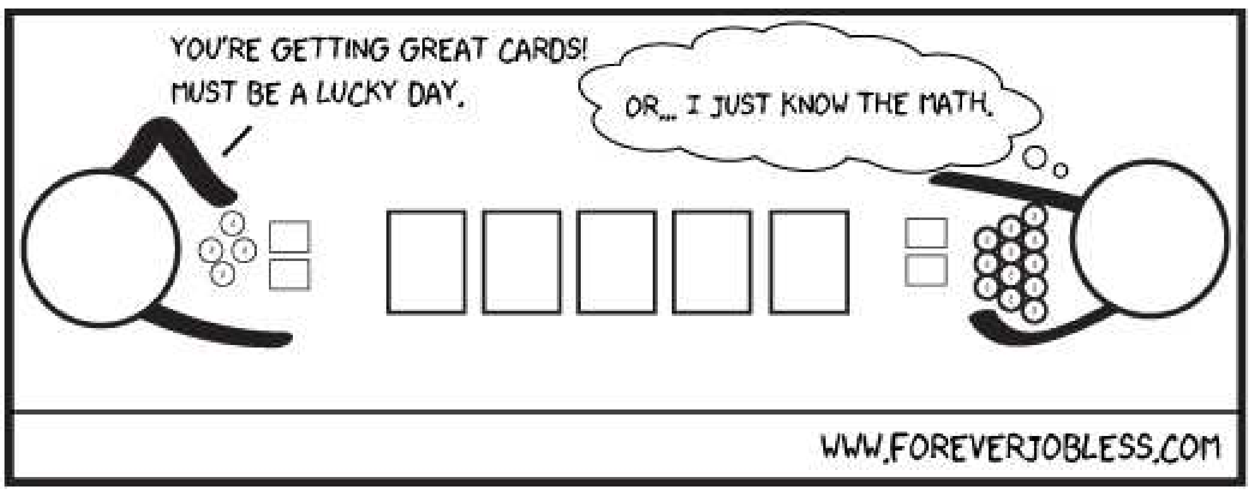
Foundations of Computer Science
Lecture 21

Deviations from the Mean

How Good is the Expectation as a Sumary of a Random Variable?
Variance: Uniform; Bernoulli; Binomial; Waiting Times.
Variance of a Sum
Law of Large Numbers: The 3-σ Rule



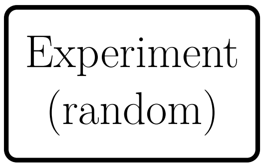
Last Time

1. Expected value of a Sum.
   - Sum of dice
   - Binomial
   - Waiting time
   - Coupon collecting.
2. Build-up expectation.
3. Expected value of a product.
4. Sum of Indicators.
   - Random arrangement of hats on heads.

Today: Deviations from the Mean

1. How well does the expected value (mean) summarize a random variable?
2. Variance.
3. Variance of a sum.
4. Law of large numbers
   - The 3-σ rule.

Probability For Analyzing a Random Experiment.



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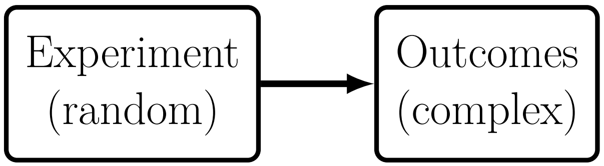

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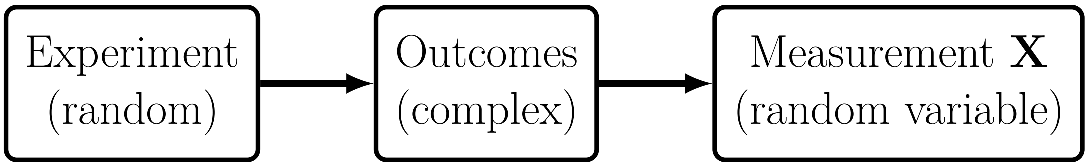

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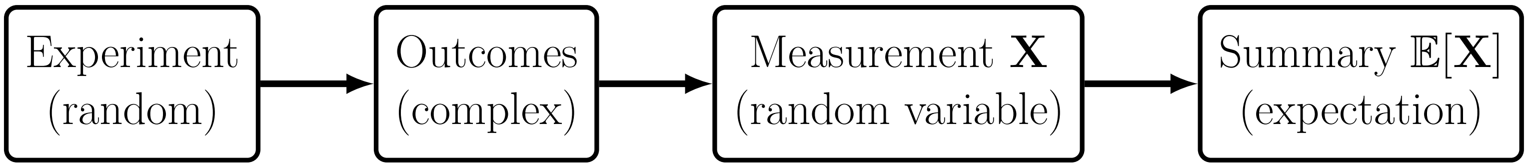

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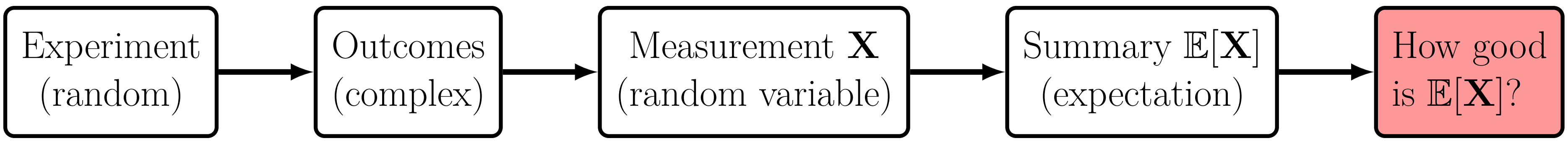

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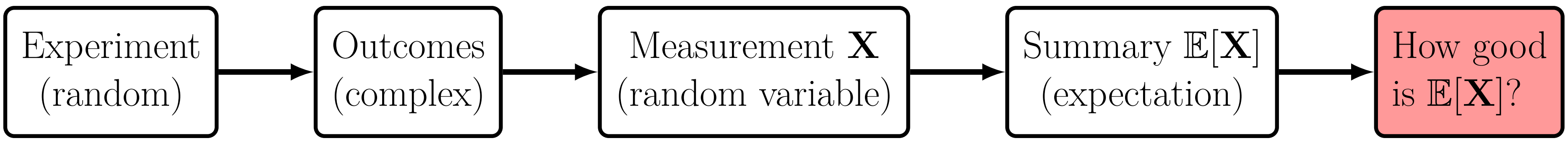

Experiment. Roll n dice and compute $\mathbf{X}$, the average of the rolls.

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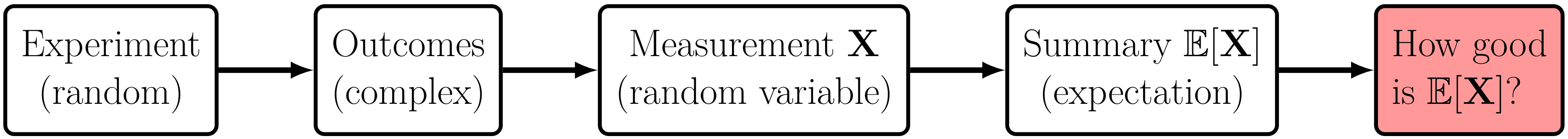

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$\mathbb{E}[\text{average}]$

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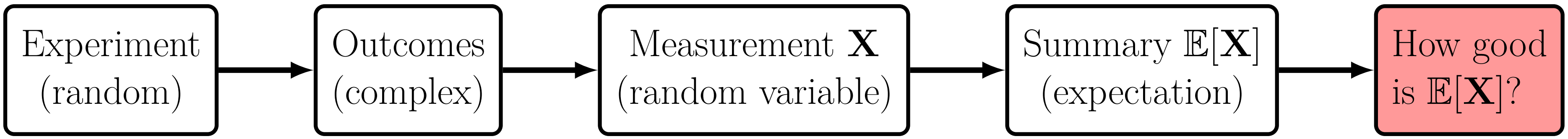

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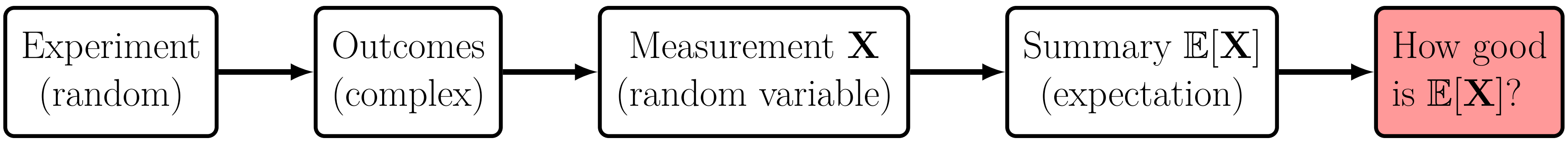

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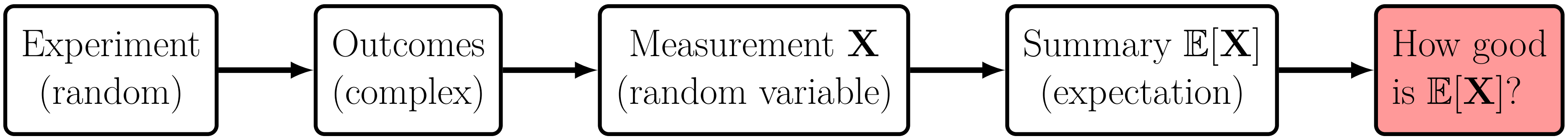

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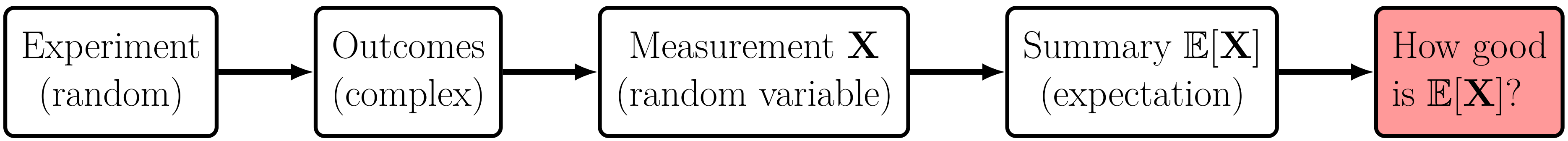

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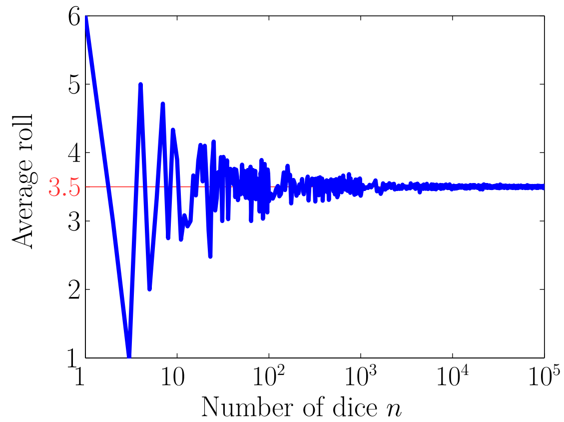
Average of n Dice



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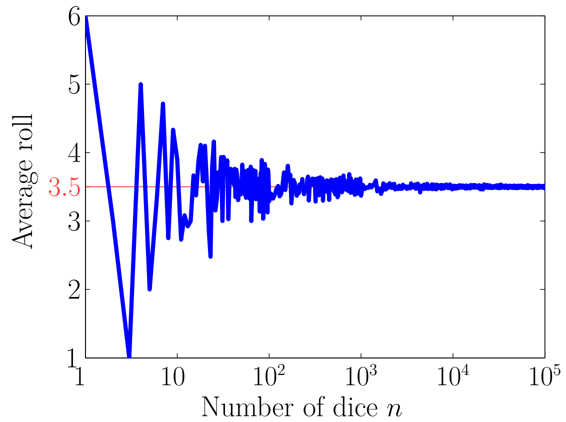

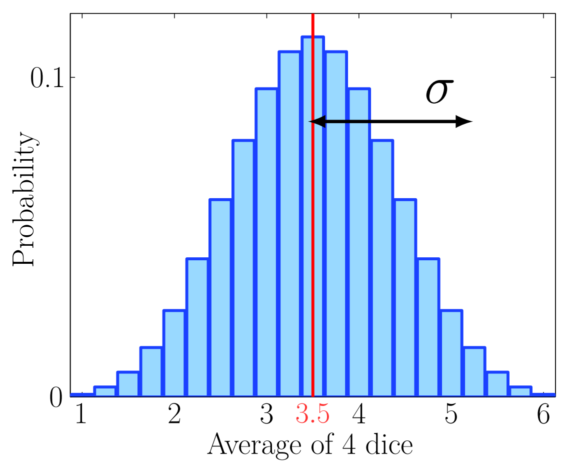

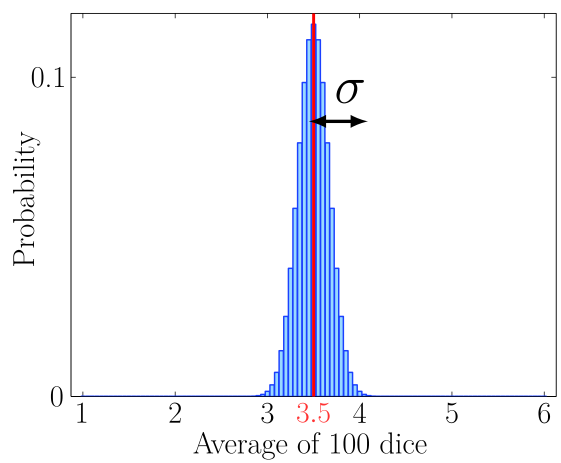

Variance: Size of the Deviations From the Mean

$\mathbf{X}$ = sum of 2 dice. $\mathbb{E}[\mathbf{X}] = 7 \leftarrow \mu(\mathbf{X})$

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$\mathbf{X}$	2	3	4	5	6	7	8	9	10	11	12	
$\boldsymbol{\Delta}$	-5	-4	-3	-2	-1	0	1	2	3	4	5	$\leftarrow \mathbf{X} - \mu$
$P_{\mathbf{X}}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	

Pop Quiz. What is $\mathbb{E}[\boldsymbol{\Delta}]$?

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$$\sigma^2 \ = \ \mathbb{E}[\boldsymbol{\Delta}^2] \ = \ \mathbb{E}[(\mathbf{X} - \mu)^2] \ = \ \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}])^2]$$

$$\sigma^2 \ = \ \mathbb{E}[\boldsymbol{\Delta}^2] \ = \ \tfrac{1}{36}\cdot 25 +$$

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Standard Deviation, σ, is the square-root of the variance,

$$\sigma = \sqrt{\mathbb{E}[\boldsymbol{\Delta}^2]} = \sqrt{\mathbb{E}[(\mathbf{X}-\mu)^2]} = \sqrt{\mathbb{E}[(\mathbf{X}-\mathbb{E}[\mathbf{X}])^2]}$$

$\sigma = \sqrt{5\frac{5}{6}} \approx 2.52$

sum of two dice rolls $= 7 \pm 2.52$.

Practice. Exercise 21.2.

Variance is a Measure of Risk

Game 1

Game 2

Variance is a Measure of Risk

Game 1

$$\mathbf{X}_1: \quad \begin{array}{ll} \text{win } \$2 & \text{probability} = \frac{2}{3}; \\ \text{lose } \$1 & \text{probability} = \frac{1}{3}. \end{array}$$

Game 2

Variance is a Measure of Risk

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Game 2

$$\mathbf{X}_2 : \quad \begin{array}{rl} \text{win } \$102 & \text{probability} = \frac{2}{3}; \\ \text{lose } \$201 & \text{probability} = \frac{1}{3}. \end{array}$$

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$$\mathbb{E}[\mathbf{X}_1] = \$1$$

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$$\begin{aligned} \sigma^2(\mathbf{X}_1) &= \tfrac{2}{3} \cdot (2-1)^2 + \tfrac{1}{3} \cdot (-1-1)^2 \\ &= 2 \end{aligned}$$

<u>Game 2</u>

$$\mathbf{X}_2 : \quad \begin{array}{ll} \text{win } \$102 & \text{probability} = \frac{2}{3}; \\ \text{lose } \$201 & \text{probability} = \frac{1}{3}. \end{array}$$

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Variance is a Measure of Risk

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$$\mathbf{X}_1 : \begin{array}{ll} \text{win \$2} & \text{probability} = \frac{2}{3}; \\ \text{lose \$1} & \text{probability} = \frac{1}{3}. \end{array}$$

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$$\begin{aligned} \mathbf{X}_1 &= 1 \pm 1.41 \\ \mathbf{X}_2 &= 1 \pm \mathbf{141} \end{aligned}$$

For a small expected profit you might risk a small loss (Game 1), not a huge loss.

A More Convenient Formula for Variance

$$\sigma^2 = \mathbb{E}[(\mathbf{X} - \mu)^2]$$

A More Convenient Formula for Variance

$$\begin{aligned} \sigma^2 &= \mathbb{E}[(\mathbf{X} - \mu)^2] \\ &= \mathbb{E}[\mathbf{X}^2 - 2\mu\mathbf{X} + \mu^2] \end{aligned}$$

← Expand $(\mathbf{X} - \mu)^2$

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Variance: $\sigma^2 = \mathbb{E}[\mathbf{X}^2] - \mu^2 = \mathbb{E}[\mathbf{X}^2] - \mathbb{E}[\mathbf{X}]^2.$

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Sum of two dice,

$$\mathbb{E}[\mathbf{X}^2] = \sum_{x=2}^{12} P_{\mathbf{X}}(x) \cdot x^2$$

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Variance: $\sigma^2 = \mathbb{E}[\mathbf{X}^2] - \mu^2 = \mathbb{E}[\mathbf{X}^2] - \mathbb{E}[\mathbf{X}]^2.$

Sum of two dice,

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\mathbb{E}[\mathbf{X}^2] &= \sum_{x=2}^{12} P_{\mathbf{X}}(x) \cdot x^2 \\
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\sigma^2 &= \mathbb{E}[(\mathbf{X}-\mu)^2] \\
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Since $\mu = 7$

$$\sigma^2 = 54\tfrac{5}{6} - 7^2 = 5\tfrac{5}{6}$$

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Theorem. Variance ≥ 0, which means $\mathbb{E}[\mathbf{X}^2] \geq \mathbb{E}[\mathbf{X}]^2$ for any random variable $\mathbf{X}$.

Variance of Uniform and Bernoulli

Uniform. We saw earlier that $\mathbb{E}[\mathbf{X}] = \frac{1}{2}(n+1)$.

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Let $\mathbf{X}$ be a Bernoulli and $\mathbf{Y} = a + \mathbf{X}$ (a is a constant):

$$\mathbf{Y} = \begin{cases} a+1 & \text{with probability } p; \\ a & \text{with probability } 1-p. \end{cases}$$

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Pop Quiz. $\mathbf{Y} = b\mathbf{X}$. Compute $\mathbb{E}[\mathbf{Y}]$ and $\sigma^2(\mathbf{Y})$.

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Theorem. Let $\mathbf{Y} = a + b\mathbf{X}$. Then,

$$\sigma^2(\mathbf{Y}) = b^2\sigma^2(\mathbf{X}).$$

Variance of a Sum

$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2$

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$(*)$ is by linearity of expectation.

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&= \underbrace{\mathbb{E}[\mathbf{X}_1^2] - \mathbb{E}[\mathbf{X}_1]^2}_{\sigma^2(\mathbf{X}_1)} + \underbrace{\mathbb{E}[\mathbf{X}_2^2] - \mathbb{E}[\mathbf{X}_2]^2}_{\sigma^2(\mathbf{X}_2)} + 2\underbrace{(\mathbb{E}[\mathbf{X}_1\mathbf{X}_2] - \mathbb{E}[\mathbf{X}_1]\,\mathbb{E}[\mathbf{X}_2])}_{0 \text{ if } \mathbf{X}_1 \text{ and } \mathbf{X}_2 \text{ are independent}}
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\end{aligned}$$

Variance of a Sum. For *independent* random variables, the variance of the sum is a sum of the variances.

Practice. Compute the variance of 1 dice roll. Compute the variance of the sum of n dice rolls.

Variance of a Sum

$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2$

$$\mathbb{E}[\mathbf{X}]^2 = \mathbb{E}[\mathbf{X}_1 + \mathbf{X}_2]^2 \overset{(*)}{=} (\mathbb{E}[\mathbf{X}_1] + \mathbb{E}[\mathbf{X}_2])^2 = \mathbb{E}[\mathbf{X}_1]^2 + \mathbb{E}[\mathbf{X}_2]^2 + 2\,\mathbb{E}[\mathbf{X}_1]\,\mathbb{E}[\mathbf{X}_2];$$

$$\mathbb{E}[\mathbf{X}^2] = \mathbb{E}[(\mathbf{X}_1 + \mathbf{X}_2)^2] = \mathbb{E}[\mathbf{X}_1^2 + \mathbf{X}_2^2 + 2\mathbf{X}_1\mathbf{X}_2] \overset{(*)}{=} \mathbb{E}[\mathbf{X}_1^2] + \mathbb{E}[\mathbf{X}_2^2] + 2\,\mathbb{E}[\mathbf{X}_1\mathbf{X}_2].$$

$(*)$ is by linearity of expectation.

$$\begin{aligned}\sigma^2(\mathbf{X}) &= \mathbb{E}[\mathbf{X}^2] - \mathbb{E}[\mathbf{X}]^2 \\ &= (\mathbb{E}[\mathbf{X}_1^2] + \mathbb{E}[\mathbf{X}_2^2] + 2\,\mathbb{E}[\mathbf{X}_1\mathbf{X}_2]) - (\mathbb{E}[\mathbf{X}_1]^2 + \mathbb{E}[\mathbf{X}_2]^2 + 2\,\mathbb{E}[\mathbf{X}_1]\,\mathbb{E}[\mathbf{X}_2]) \\ &= \underbrace{\mathbb{E}[\mathbf{X}_1^2] - \mathbb{E}[\mathbf{X}_1]^2}_{\sigma^2(\mathbf{X}_1)} + \underbrace{\mathbb{E}[\mathbf{X}_2^2] - \mathbb{E}[\mathbf{X}_2]^2}_{\sigma^2(\mathbf{X}_2)} + 2\underbrace{(\mathbb{E}[\mathbf{X}_1\mathbf{X}_2] - \mathbb{E}[\mathbf{X}_1]\,\mathbb{E}[\mathbf{X}_2])}_{0 \text{ if } \mathbf{X}_1 \text{ and } \mathbf{X}_2 \text{ are independent}}\end{aligned}$$

Variance of a Sum. For *independent* random variables, the variance of the sum is a sum of the variances.

Practice. Compute the variance of 1 dice roll. Compute the variance of the sum of n dice rolls.

Example. The Variance of the Binomial (sum of *independent* Bernoullis)

$\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n$ (sum of *independent* Bernoullis), and $\sigma^2(\mathbf{X}_i) = p(1-p)$

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$$\sigma^2(\text{Binomial}) = \sigma^2(\mathbf{X}_1) + \cdots + \sigma^2(\mathbf{X}_n) = p(1-p) + \cdots + p(1-p) = np(1-p).$$

3-σ Rule: $\mathbf{X} = \mu(\mathbf{X}) \pm \sigma(\mathbf{X})$

3-σ Rule. For *any* random variable $\mathbf{X}$, the chances are at least (about) 90% that

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In (a) we used Markov's Inequality. ■

To get the 3-σ rule, use Chebyshev's Inequality with $t = 3$.

Law of Large Numbers

Expectation of the average of n dice:

$$\mathbb{E}[\text{average}] = \mathbb{E}[\tfrac{1}{n} \times \text{sum}] = \tfrac{1}{n} \times \mathbb{E}[\text{sum}] = \tfrac{1}{n} \times n \times 3\tfrac{1}{2}$$

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