Foundations of Computer Science
Lecture 16

Conditional Probability

Updating a Probability when New Information Arrives
Conditional Probability Traps
Law of Total Probability

Last Time

1. Outcome-tree method for computing probability.
2. Probability and sets.
   - Probability space.
   - Event is a subset of outcomes.
   - Can get complex events using set (logical) operations.
3. Uniform probability space
   - Toss 10 coins. Each sequence (e.g. HTHHHTTTHH) has equal probability.
   - Roll 3 dice. Each sequence (e.g. (2,4,5)) has equal probability.
   - Probability of event $\sim$ event size.
4. Infinite probability space.
   - Toss a coin until you get heads (possibly never ending).

Today: Conditional Probability

1. New information changes a probability.

2. Definition of conditional probability from regular probability.

3. Conditional probability traps
   - Sampling bias.
   - Transposed conditional.

4. Law of total probability.
   - Probabilistic case-by-case analysis.

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$$\text{Probability of flu } \textit{given} \text{ fever} : \mathbb{P}[\text{flu} \mid \text{fever}] \approx 0.4.$$

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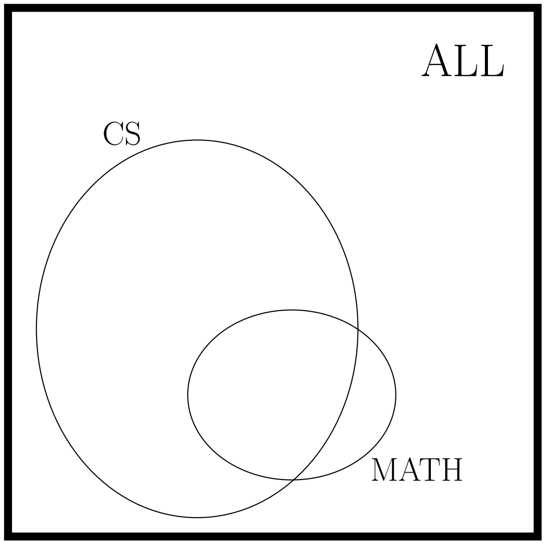
3. Roommie has flu (more new information). Flu for sure, take counter-measures.

$$\text{Probability of flu } \textit{given} \text{ fever and roommie flu} : \mathbb{P}[\text{flu} \mid \text{fever AND roommie flu}] \approx 1.$$

Pop Quiz. Estimate these probabilities:
$\mathbb{P}[\text{Humans alive tomorrow}]$, $\mathbb{P}[\text{No Sun tomorrow}]$, $\mathbb{P}[\text{Humans alive tomorrow} \mid \text{No Sun tomorrow}]$.

CS, MATH and Dual CS-MATH Majors

5,000 students: **1,000** CS; **100** MATH; **80** dual MATH-CS.



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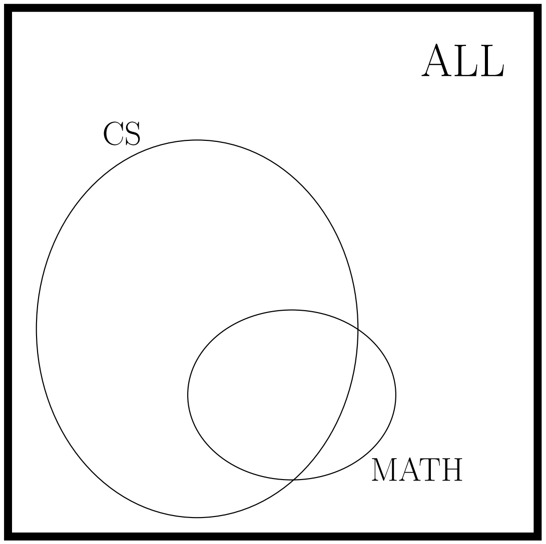

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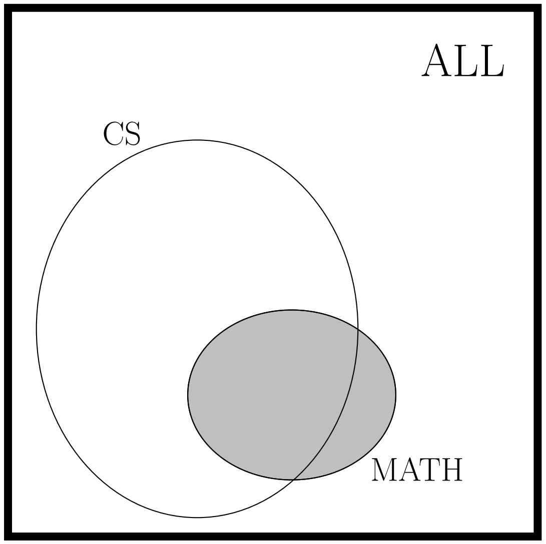

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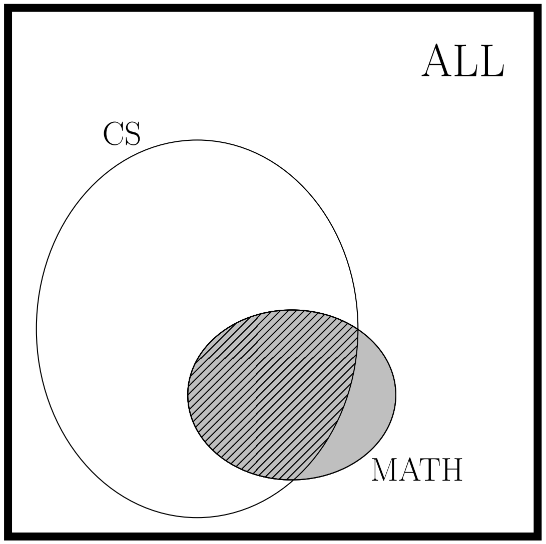

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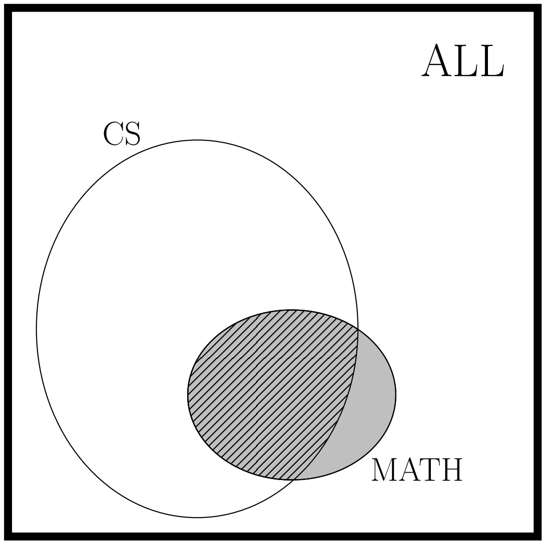

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$$\mathbb{P}[\text{CS} \mid \text{MATH}] = \frac{|\text{CS} \cap \text{MATH}|}{|\text{MATH}|} = \frac{80}{100} = 0.8.$$

MATH students are 4 times more likely to be CS majors than a random student.

Pop Quiz. What is $\mathbb{P}[\text{MATH} \mid \text{CS}]$? What is $\mathbb{P}[\text{CS} \mid \text{CS OR MATH}]$? **Exercise 16.2.**

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Chances of Rain Given Clouds

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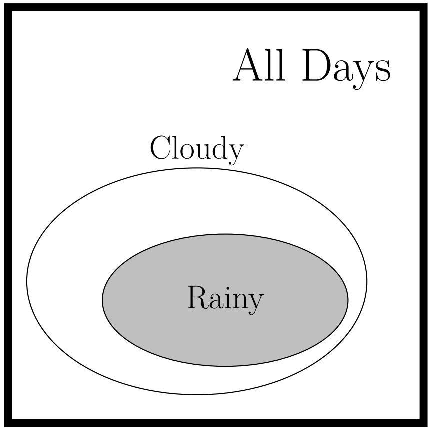

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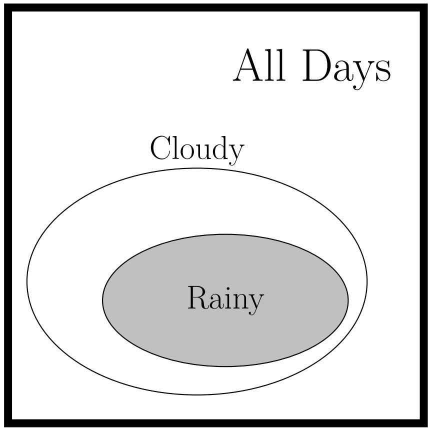
$$\mathbb{P}[\text{Rain} \mid \text{Clouds}] = \frac{\mathbb{P}[\text{Rain}]}{\mathbb{P}[\text{Clouds}]} = \frac{\frac{1}{7}}{\frac{1}{5}} = \frac{5}{7}.$$



5-times more likely to rain on a cloudy day than on a random day.

Crucial first step: identify the conditional probability. What is the "new information"?

$\mathbb{P}$[Sum of 2 Dice is 10 | Both are Odd]

Two dice have both rolled odd. What are the chances the sum is 10?

$$\mathbb{P}[\text{Sum is 10} \mid \text{Both are Odd}] = \frac{\mathbb{P}[(\text{Sum is 10}) \text{ AND } (\text{Both are Odd})]}{\mathbb{P}[\text{Both are Odd}]}$$

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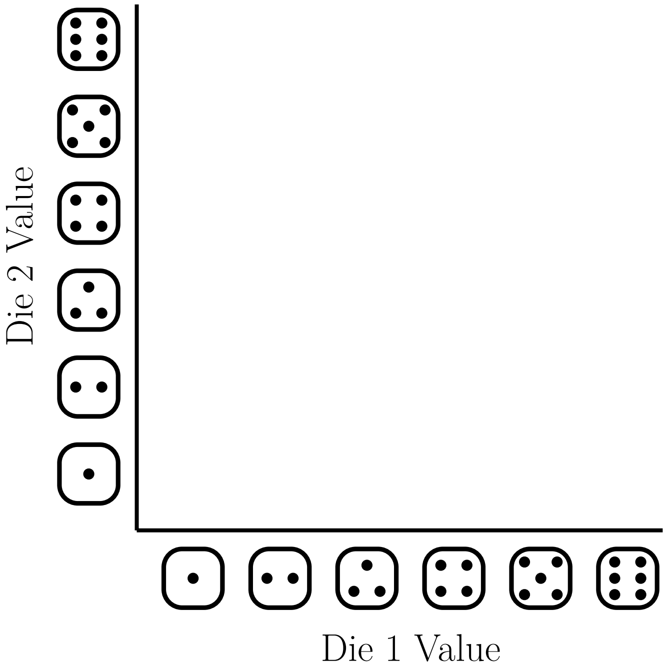

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Die 1 Value

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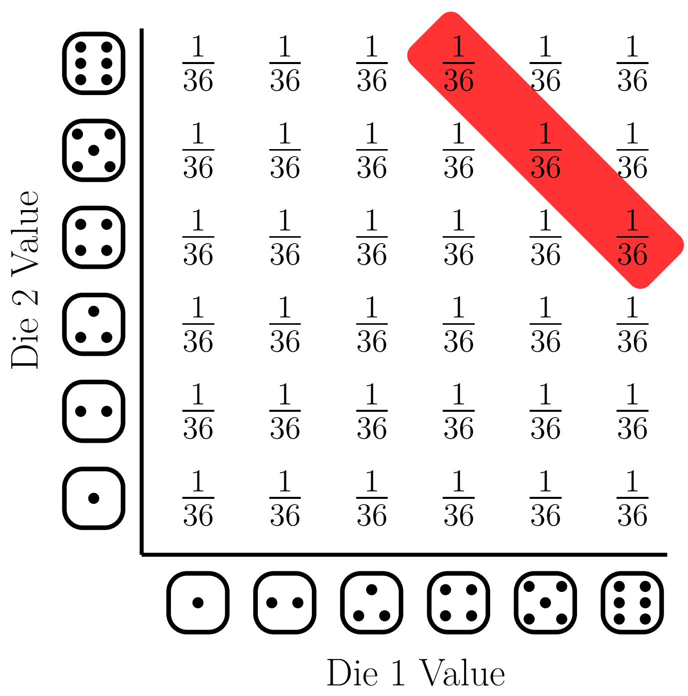

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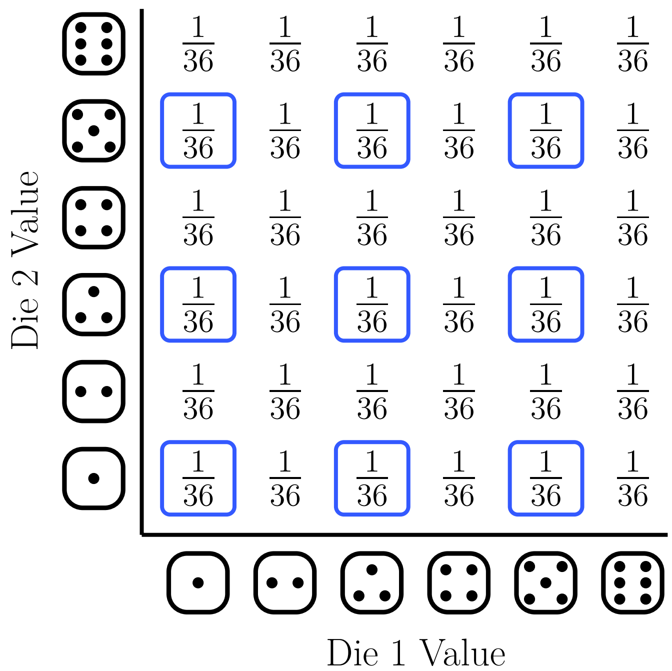

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2. $\mathbb{P}[\text{Both are Odd}] = \frac{9}{36} = \frac{1}{4}$.

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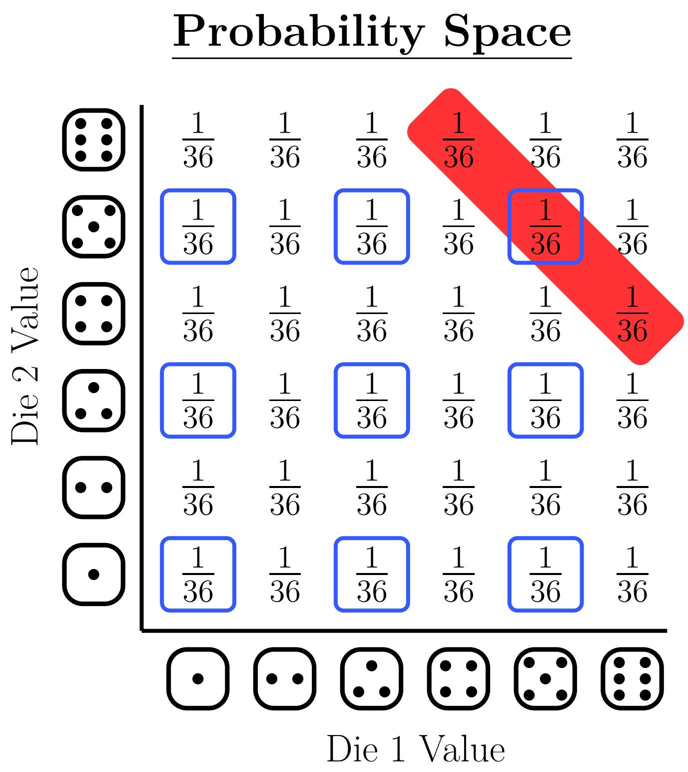

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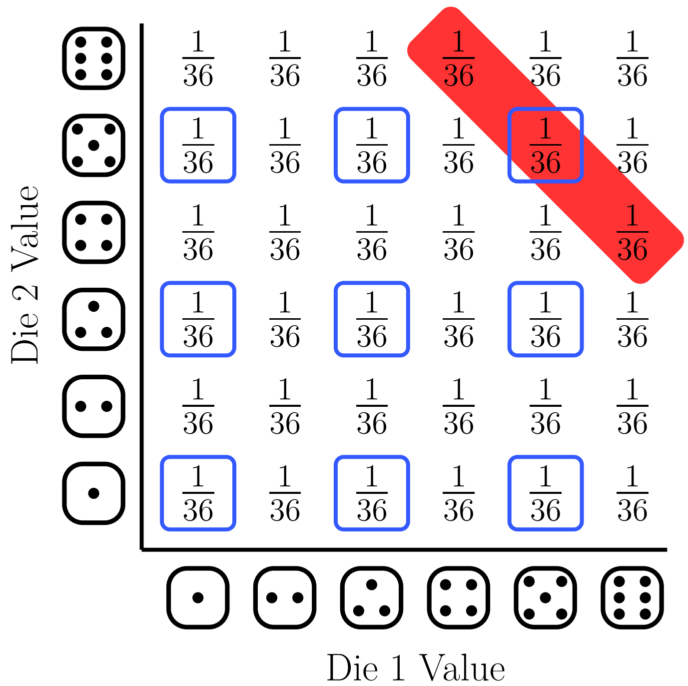

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4. $\mathbb{P}[\text{Sum is 10} \mid \text{Both are Odd}] = \frac{1}{36} \div \frac{1}{4} = \frac{1}{9}$.

Pop Quiz. Compute $\mathbb{P}$[Both are Odd | Sum is 10]. Compare with $\mathbb{P}$[Sum is 10 | Both are Odd].

Computing a Conditional Probability

1: Identify that you need a conditional probability $\mathbb{P}[A \mid B]$.

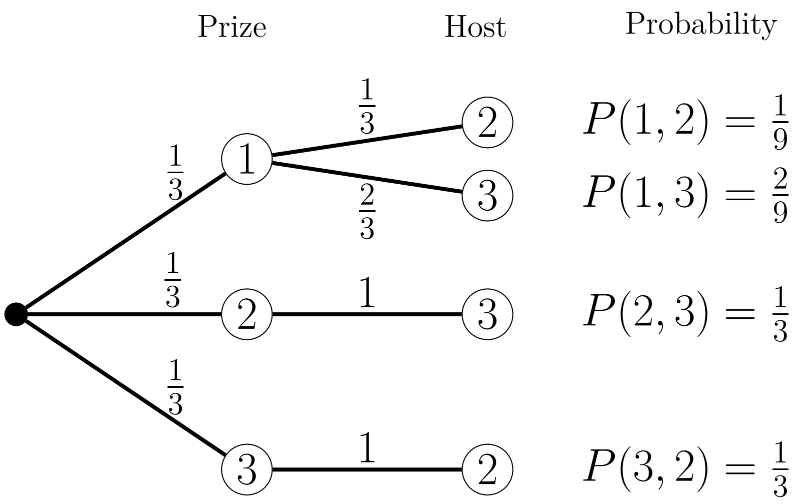
2: Determine the probability space $(\Omega, P(\cdot))$ using the outcome-tree method.

3: Identify the events A and B appearing in $\mathbb{P}[A \mid B]$ as subsets of Ω.

4: Compute $\mathbb{P}[A \cap B]$ and $\mathbb{P}[B]$.

5: Compute $\mathbb{P}[A \mid B] = \dfrac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$.

Monty Prefers Door 3



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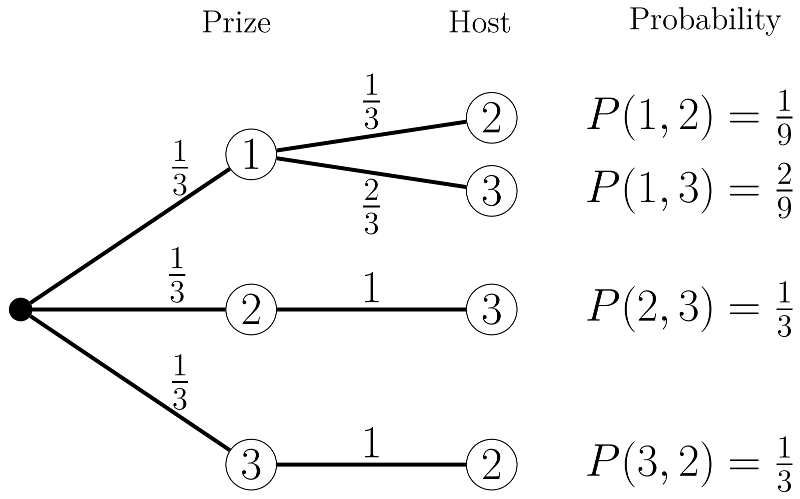

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Winning outcomes: (2,3) or (3,2).

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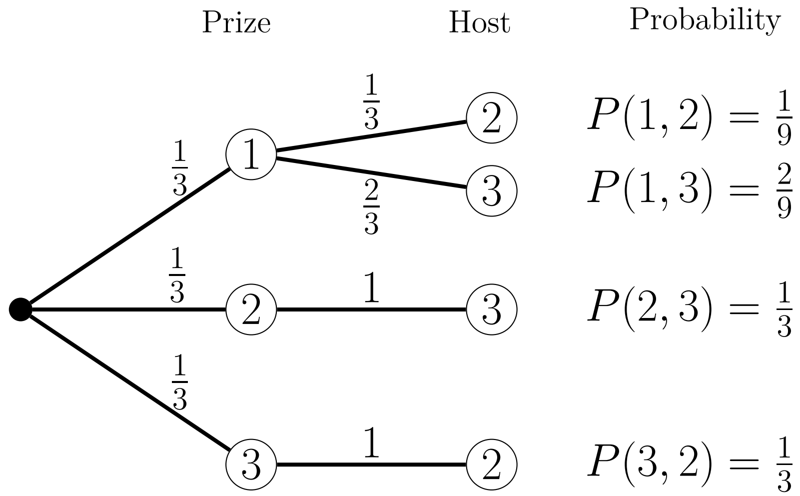

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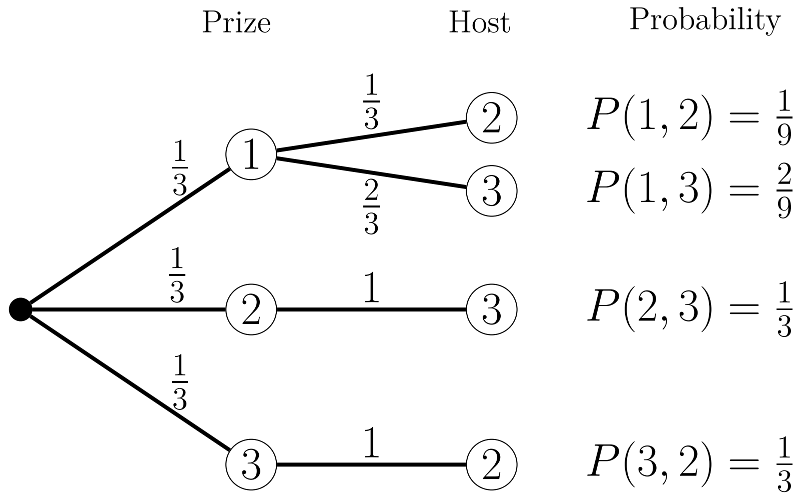

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$$\begin{aligned}\mathbb{P}[\text{Win}|\text{Monty opens Door 3}] &= \frac{\mathbb{P}[\text{Win AND Monty opens Door 3}]}{\mathbb{P}[\text{Monty opens Door 3}]} \\ &= \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{9}} \\ &= \tfrac{3}{4}.\end{aligned}$$

Your chances improved from $\frac{2}{3}$ to $\frac{3}{4}$!

A Pair of Boys

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What is the probability of two boys?

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New information:

1. Ayfos has at least one boy.
2. Ifar's older child is a boy.
3. One day you met Need on a walk with a boy.
4. Niaz is Clingon. Clingons always take a son on a walk if possible. One day, you met Niaz on a walk with a boy.

Now, what is the probability of two boys in each case?

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It's the same question in each case, but with slightly different additional information.
You need conditional probabilities.

Conditional Probability Traps

These four probabilities are all different.

$$\mathbb{P}[A] \qquad \mathbb{P}\left[A \mid B\right] \qquad \mathbb{P}\left[B \mid A\right] \qquad \mathbb{P}\left[A \text{ AND } B\right]$$

Don't use one when you should use another.

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Sampling Bias: Using $P[A]$ instead of $P[A \mid B]$

$$\mathbb{P}[\text{Voter will vote Republican}] \approx \tfrac{1}{2}.$$

Ask Apple™ to call up i-Phone™ users to see how they will vote.

$$\mathbb{P}[\text{Voter will vote Republican} \mid \text{Voter has an i-Phone}] \gg \tfrac{1}{2}. \qquad \text{(Why?)}$$

This has trapped many US election-pollers. For a famous example, Google™ "Dewey Defeats Truman."

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Transposed Condtional: Using $P[B \mid A]$ instead of $P[A \mid B]$

Famous Lombard study on the riskiest profession: **Student!** Lombard confused:

$$\mathbb{P}[\text{Student} \mid \text{Die Young}] \qquad \text{with} \qquad \mathbb{P}[\text{Die Young} \mid \text{Student}]$$

The LAME Test and Transposed Conditionals

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It's *wrong* to look at $\mathbb{P}$[positive | not LAME]. We need $\mathbb{P}$[not LAME | positive].

The LAME Test and Transposed Conditionals

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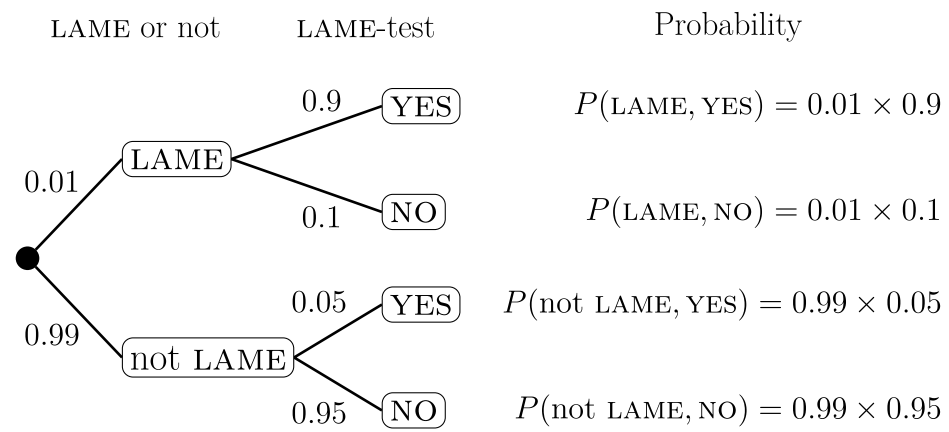
If you are not LAME, the test makes a mistake in only 5% of cases.

You get tested positive. What are the chances you are LAME?

If you are not LAME, the test wouldn't make a mistake. So you are likely LAME.

It's *wrong* to look at $\mathbb{P}[\text{positive} \mid \text{not LAME}]$. We need $\mathbb{P}[\text{not LAME} \mid \text{positive}]$.



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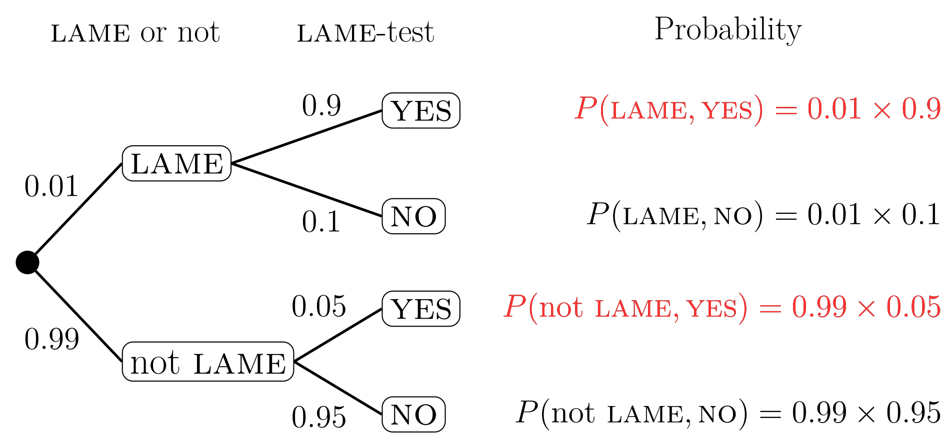
It's *wrong* to look at $\mathbb{P}$[positive | not LAME]. We need $\mathbb{P}$[not LAME | positive].

$$\mathbb{P}\left[\text{not LAME} \mid \text{YES}\right] = \frac{\mathbb{P}[\text{not LAME AND YES}]}{\mathbb{P}[\text{YES}]}$$



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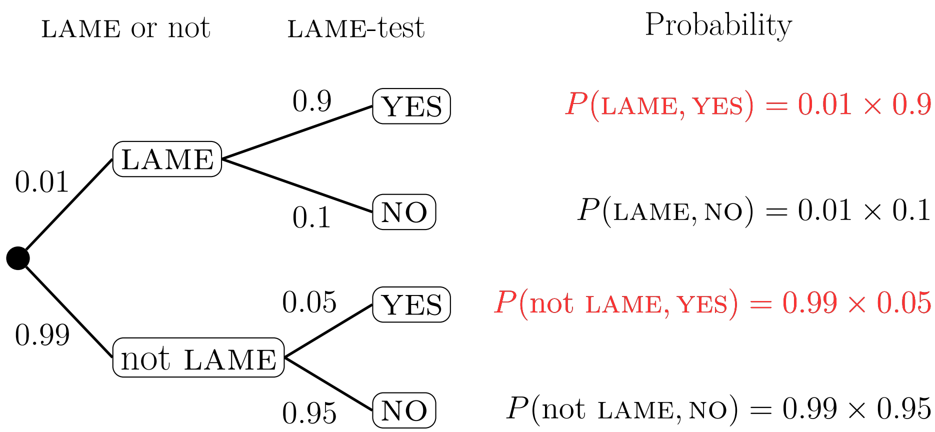
$$\mathbb{P}\left[\text{not LAME} \mid \text{YES}\right] = \frac{\mathbb{P}[\text{not LAME AND YES}]}{\mathbb{P}[\text{YES}]}$$

$$= \frac{0.99 \times 0.05}{0.99 \times 0.05 + 0.9 \times 0.01}$$

$$\approx 85\%.$$



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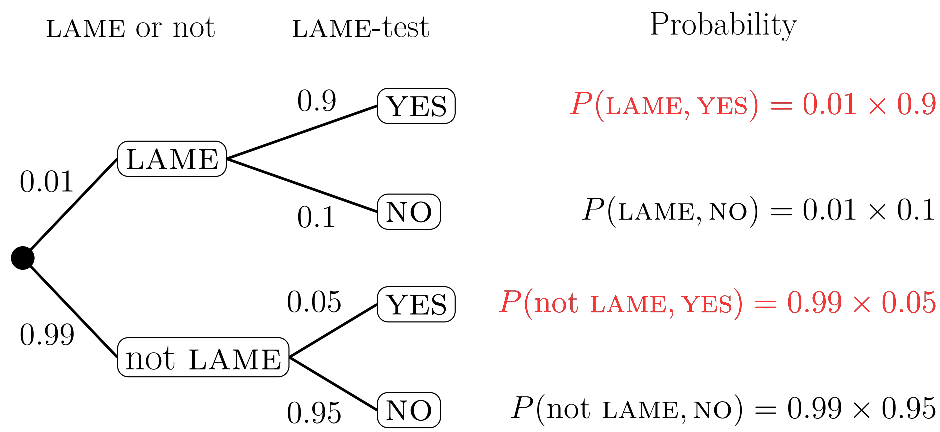
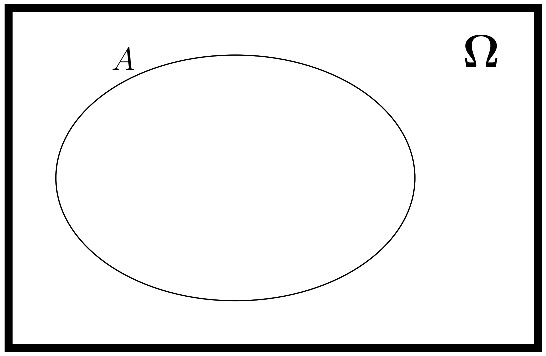


The (accurate) test says YES but the chances are 85% that you are not LAME!

- You are LAME (rare) plus the test was right (likely)
- You are not LAME (very likely) plus the test got it wrong (rare). Wins!

Total Probability: Case by Case Probability

Two types of outcomes in any event A:



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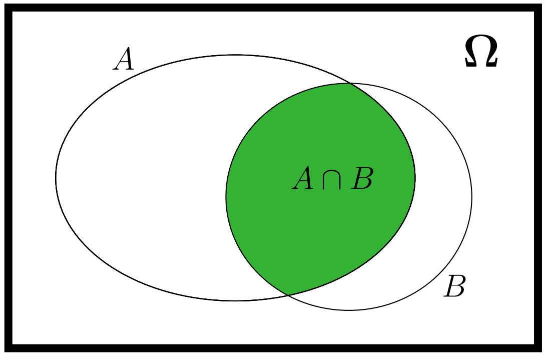

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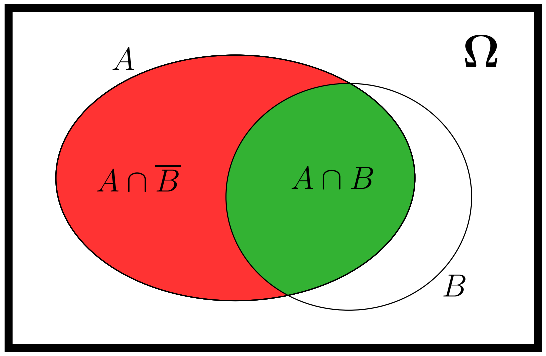

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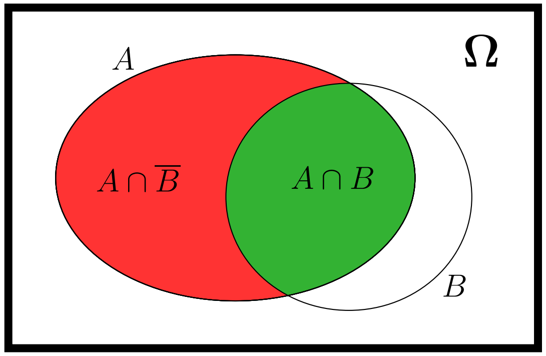
$$\mathbb{P}[A] = \mathbb{P}[A \cap B] + \mathbb{P}[A \cap \overline{B}]. \qquad (*)$$

(Similar to sum rule from counting.)



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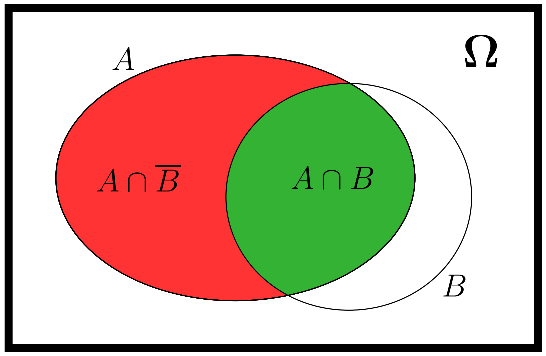

From the definition of conditional probability:

$$\begin{aligned}\mathbb{P}[A \cap B] &= \mathbb{P}[A \text{ AND } B] = \mathbb{P}[A \mid B] \times \mathbb{P}[B];\\ \mathbb{P}[A \cap \overline{B}] &= \mathbb{P}[A \text{ AND } \overline{B}] = \mathbb{P}[A \mid \overline{B}] \times \mathbb{P}[\overline{B}].\end{aligned}$$

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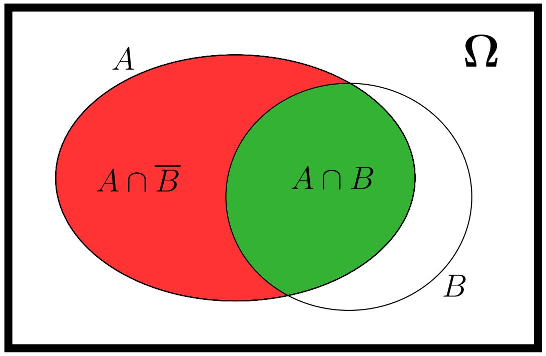

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Plugging these into $(*)$, we get a **FUNDAMENTAL** result for case by case analysis:

Law of Total Probability

$$\mathbb{P}[A] = \mathbb{P}[A \mid B] \cdot \mathbb{P}[B] + \mathbb{P}[A \mid \overline{B}] \cdot \mathbb{P}[\overline{B}].$$

(Weight conditional probabilities for each case by probabilities of each case and add.)

Three Coins: Two Are Fair, One is 2-Headed

Pick a random coin and flip. What is the probability of H?

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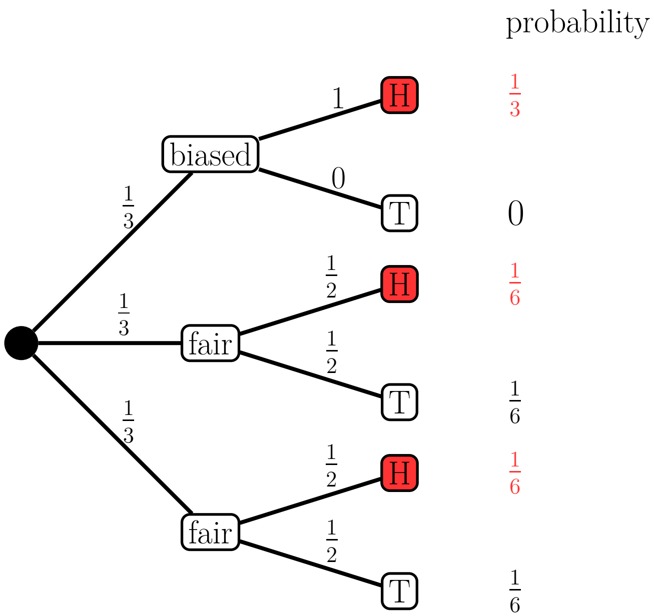
Outcome-Tree Method



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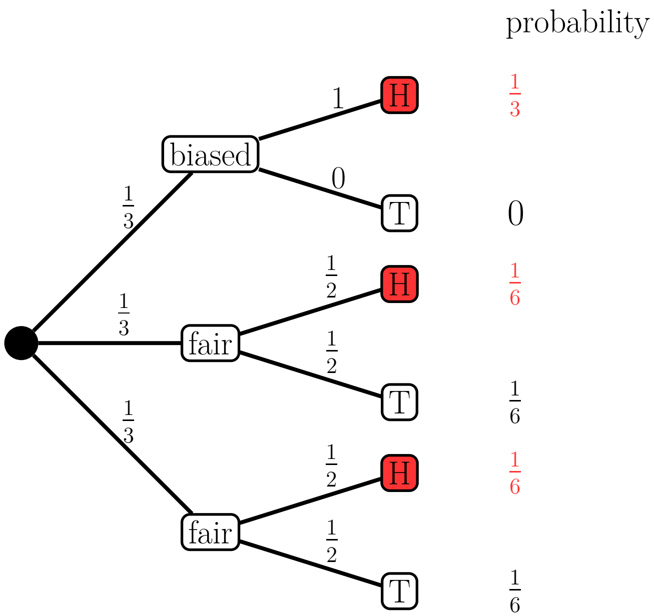

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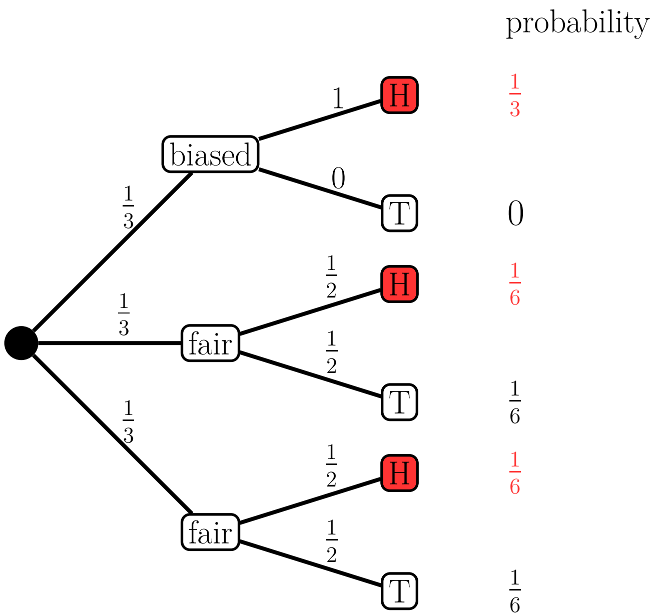

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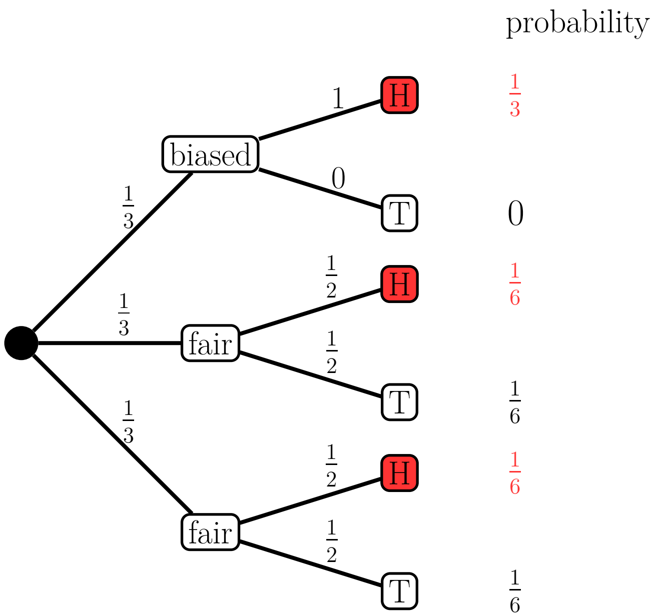

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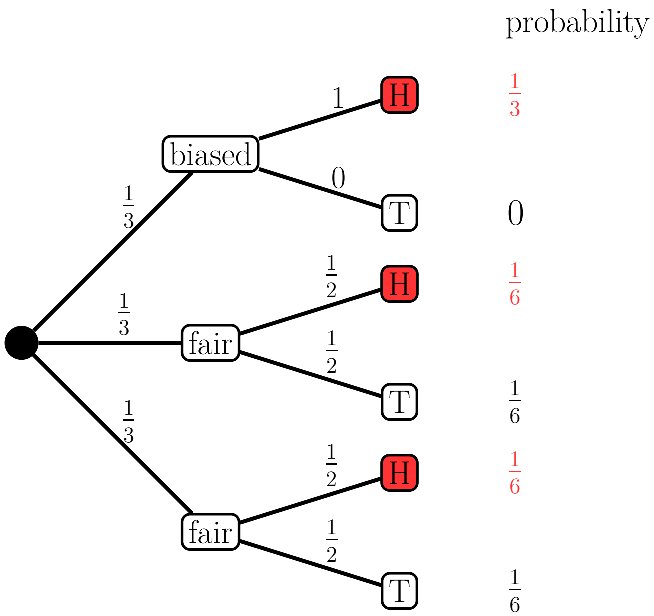

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Exercise. A box has 10 coins: 6 fair and 4 biased (probability of heads $\frac{2}{3}$). What is $\mathbb{P}[2 \text{ heads}]$ in each case?

- (a) Pick a single random coin and flip it 3 times.
- (b) Flip 3 times. For each flip, pick a random coin, flip it and then put the coin back.

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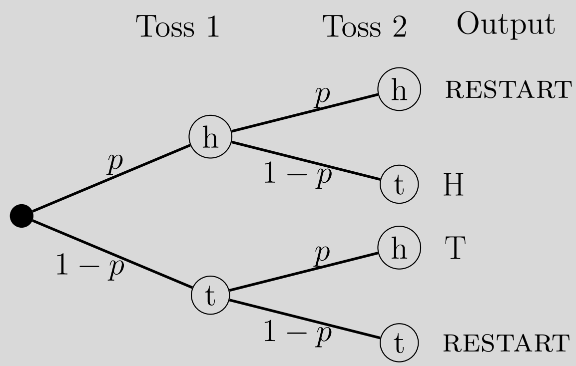

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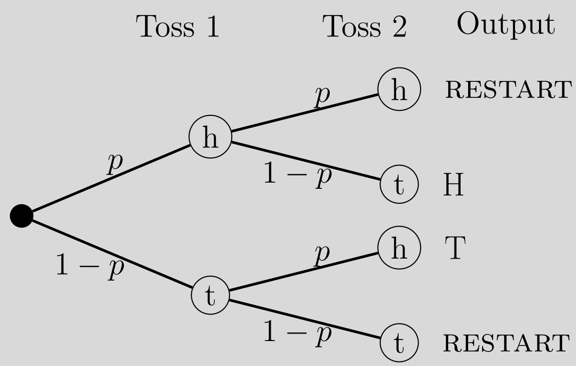

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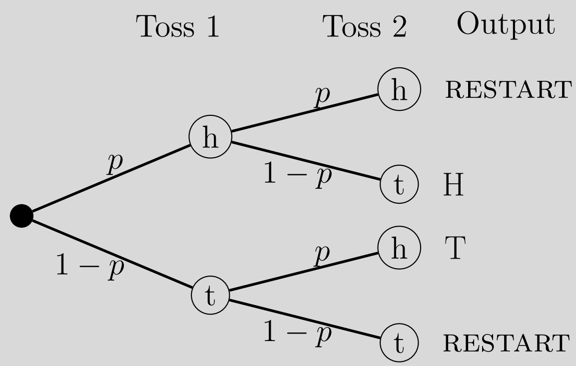

By the law of total probability (3 cases),

$$\mathbb{P}[\text{H}] = \underset{\mathbb{P}[\text{H}]}{\underset{\uparrow}{\mathbb{P}[\text{H} \mid \text{RESTART}]}} \cdot \underset{p^2+(1-p)^2}{\underset{\uparrow}{\mathbb{P}[\text{RESTART}]}} + \underset{1}{\underset{\uparrow}{\mathbb{P}[\text{H} \mid \text{'ht'}]}} \cdot \underset{p(1-p)}{\underset{\uparrow}{\mathbb{P}[\text{'ht'}]}} + \underset{0}{\underset{\uparrow}{\mathbb{P}[\text{H} \mid \text{'th'}]}} \cdot \underset{p(1-p)}{\underset{\uparrow}{\mathbb{P}[\text{'th'}]}}$$

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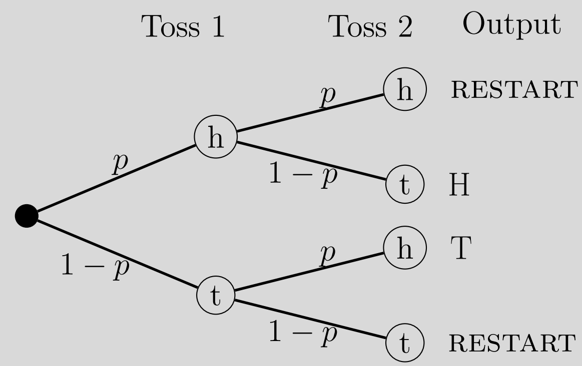

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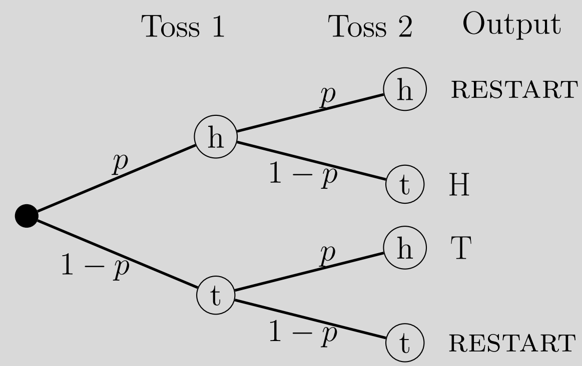

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Solve for $\mathbb{P}[\text{H}]$

$$\mathbb{P}[\text{H}] = \frac{p(1-p)}{1-(p^2+(1-p)^2)} = \frac{p(1-p)}{2p-2p^2} = \frac{p(1-p)}{2p(1-p)} = \frac{1}{2}.$$

(You can also solve this problem using an infinite outcome tree and computing an infinite sum.)