I. Eight pawns on a chessboard

(a) Probability that all eight pawns lie in a straight line

Since the eight pawns are *identical* and are placed on eight distinct squares chosen uniformly at random from the 64 squares of an 8×8 chessboard. Hence, the total number of ways to place the pawns is

$$\binom{64}{8}.$$

A "straight line" on a chessboard includes:

- Any of the 8 rows (each row has exactly 8 squares),
- Any of the 8 columns (each column has exactly 8 squares),
- Either of the 2 main diagonals (each contains exactly 8 squares).

Thus, there are $8 + 8 + 2 = 18$ distinct lines of length 8. On any such line of 8 squares, there is exactly one way to place all 8 identical pawns (namely, occupy all the squares on that line). Therefore, the number of favorable placements (where the pawns lie in a single straight line) is 18. Consequently, the probability is

$$P(\text{all in a straight line}) = \frac{18}{\binom{64}{8}}.$$

(b) Probability that no two pawns share a row or a column

We want exactly one pawn in each row and one in each column. Equivalently, for each of the 8 rows, we must choose exactly one of the 8 columns so that no two chosen columns coincide. The number of ways to do this (for identical pawns) is precisely the number of permutations of 8 distinct elements, i.e. 8!.

Hence, the probability that no two pawns lie in the same row or column is

$$P(\text{no two in the same row or column}) = \frac{8!}{\binom{64}{8}}.$$

II. Drawing from 10 envelopes

Let the distinct amounts in the envelopes be labeled

$$a_1 < a_2 < \cdots < a_{10}.$$

We want to calculate the probability that the amount in the first drawn envelope is larger than the amount in the second drawn envelope. Denote by X the amount in the first envelope and by Y the amount in the second envelope.

$$\mathbb{P}(X > Y) \ = ?$$

For *each* of these pairs of amounts (a_i, a_j) with $i \neq j$, there are 2 ways they could be drawn:

$$(X = a_i, Y = a_j) \quad \text{or} \quad (X = a_j, Y = a_i).$$

Thus, for each pair of amounts, the chance of having $X > Y$ is exactly 1/2.

$$\mathbb{P}(X > Y) \ = \ \frac{1}{2}.$$

Conclusion:
Because the probability of "the first drawn envelope having the higher amount" is exactly $\frac{1}{2}$, there is no advantage to drawing first or second.

III. An urn has m blue balls and n red balls. You randomly pick the balls one by one and lay them in a line. What is the probability that the last ball is red?

To solve this problem, we must first find the number of ways we can pull the balls out of the urn. Notice this is equivalent to finding the number of sequences of $m+n$ balls where we choose m balls to be blue. This is given by $\binom{m+n}{m}$.

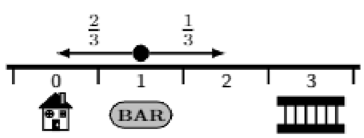
We also need the number of sequences of $m+n$ balls where the last ball is fixed as red. This is given by the expression $\binom{m+n-1}{m}$ since after we fix the last ball to be red, we are choosing m blue balls in a sequence of $m+n-1$ balls.

We can now compute the probability that the last ball is red as follows:

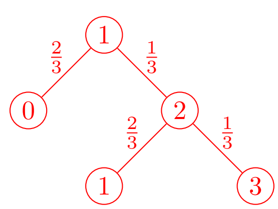
$$\begin{aligned}\mathbb{P}[\text{Last ball is red}] &= \frac{\binom{m+n-1}{m}}{\binom{m+n}{m}} \\ &= \frac{(m+n-1)!}{(m+n)!} \cdot \frac{m!n!}{m!(n-1)!} \\ &= \boxed{\frac{n}{m+n}}\end{aligned}$$

This is equivalent to the probability of just picking a single red ball from the urn.

IV. A drunk leaves the bar at position 1, and takes random steps: left (L) with probability $\frac{2}{3}$ or right (R) with probability $\frac{1}{3}$. What is the probability the drunk reaches home (at position 0) before reaching the lockup (at position 3)?



Let p be the probability that the drunk reaches home before reaching the lockup. Let's draw an outcome tree to see where the drunk can end up after two steps.



Notice that after two steps, the drunk is either at home, at the lockup, or at their original position. If the drunk returned to their original position after two steps, then the probability that they reach home is given p. This gives us an equation for p as follows:

$$\begin{aligned}p &= \frac{2}{3} + \frac{1}{3}\left(\frac{2}{3}p + \frac{1}{3} \cdot 0\right) \\ 9p &= 6 + 2p \\ p &= \frac{6}{7}\end{aligned}$$

Therefore, the probability of the drunk reaching home before the lockup is $\boxed{\frac{6}{7}}$.

V. A parent picks a boys name as Sue with probability $0 < \beta < 1$. Monica has two children with different names. What is the probability Monica has two boys if:

(a) Monica has a boy.
- This excludes the possibility of GG, leaving BG, BB, GB
- Thus the probability is 1/3.

(b) Monica has a boy named Sue.
- P[boy], P[girl] = 0.5
- P[boy named Sue] = $0.5 * \beta$
- Given choice is $B_{\mathrm{s}}B, BB_{\mathrm{s}}, B_{\mathrm{s}}G, GB_{\mathrm{s}}$
- $B_sB = 0.25 * \beta$
- $BB_s = 0.25 * \beta$
- $B_sG = 0.25 * \beta$
- $GB_s = 0.25 * \beta$
- Thus two boys: $\dfrac{0.5\beta}{\beta} = \dfrac{1}{2}$

(c) Monica does not have a boy named Sue.
- P[boy named not Sue] = $0.5 * (1 - \beta)$
- Four possibilities, BB, BG, GB, GG, while all B's name is not Sue.
- $BB = 0.25 * (1 - \beta)^2$
- $BG = 0.25 * (1 - \beta)$
- $GB = 0.25 * (1 - \beta)$
- $GG = 0.25$
- Thus P: $\dfrac{(1-\beta)^2}{(1-\beta)^2 + 2(1-\beta) + 1}$

Problem VI

One out of n coins is 2-headed. A random coin is picked and flipped k times. All flips were H.

(a) What is the probability the coin flipped is 2-headed?

Define the following events:

- A: The selected coin is 2-headed
- B: All k flips show heads

We want to find $P(A|B)$, the probability that the coin is 2-headed given that all flips showed heads. We use Bayes' theorem:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Let's calculate each term:

- $P(A) = \frac{1}{n}$ (probability of picking the 2-headed coin)
- $P(B|A) = 1$ (probability of getting k heads when flipping a 2-headed coin)
- $P(B) = P(B|A) \cdot P(A) + P(B|\overline{A}) \cdot P(\overline{A})$ (Law of Total Probability)

For a fair coin: $P(B|\overline{A}) = \left(\frac{1}{2}\right)^k$ since each flip has a $\frac{1}{2}$ probability of being heads.
Also, $P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{n} = \frac{n-1}{n}$
So:

$$\begin{aligned} P(B) &= 1 \cdot \frac{1}{n} + \left(\frac{1}{2}\right)^k \cdot \frac{n-1}{n} \\ &= \frac{1}{n} + \frac{n-1}{n} \cdot \left(\frac{1}{2}\right)^k \end{aligned}$$

Now we can calculate $P(A|B)$:

$$\begin{aligned} P(A|B) &= \frac{P(B|A) \cdot P(A)}{P(B)} \\ &= \frac{1 \cdot \frac{1}{n}}{\frac{1}{n} + \frac{n-1}{n} \cdot \left(\frac{1}{2}\right)^k} \\ &= \frac{1}{1 + (n-1) \cdot \left(\frac{1}{2}\right)^k} \end{aligned}$$

Therefore, the probability that the coin flipped is 2-headed given that all k flips were heads is:

$$P(A|B) = \frac{1}{1 + (n-1) \cdot 2^{-k}}$$

(b) For $n = 10^6$, how high should one pick k to be 99.9% sure the 2-headed coin was flipped?

We want $P(A|B) \geq 0.999$. Using our formula from part (a):

$$\frac{1}{1 + (10^6 - 1) \cdot 2^{-k}} \geq 0.999$$

Solve for k:

$$\begin{aligned}
1 + (10^6 - 1) \cdot 2^{-k} &\leq \frac{1}{0.999} \\
(10^6 - 1) \cdot 2^{-k} &\leq \frac{1}{0.999} - 1 \\
(10^6 - 1) \cdot 2^{-k} &\leq \frac{1 - 0.999}{0.999} \\
(10^6 - 1) \cdot 2^{-k} &\leq \frac{0.001}{0.999} \approx 0.001001 \\
2^{-k} &\leq \frac{0.001001}{10^6 - 1} \approx \frac{0.001001}{10^6} = 1.001 \times 10^{-9}
\end{aligned}$$

Taking $\log_2$ of both sides:

$$\begin{aligned}
-k &\leq \log_2(1.001 \times 10^{-9}) \\
k &\geq -\log_2(1.001 \times 10^{-9})
\end{aligned}$$

Using the fact that $10^{-9} \approx 2^{-30}$ and $1.001 \approx 2^{0.0014}$, we have:

$$\begin{aligned}
k &\geq -\log_2(2^{0.0014} \times 2^{-30}) \\
k &\geq -\log_2(2^{-30+0.0014}) \\
k &\geq -(30 + 0.0014) \\
k &\geq 30 - 0.0014 \\
k &\approx 30
\end{aligned}$$

Therefore, k should be at least 30 to be 99.9% sure that the 2-headed coin was flipped.

Problem VII

Cards with distinct values $v_1, \ldots, v_m$ are dealt in random order. The kth card is largest among the cards already dealt. What is the probability it is the largest in the pack?

Define the event:

- A: The kth card is the largest value in the entire pack
- B: The kth card is the largest among the first k cards

We want to find $P(A|B)$, the probability that the kth card is the largest in the entire pack given that it's the largest among the first k cards.

If the kth card is the largest in the entire pack, then it must also be the largest among the first k cards. So $A \subseteq B$, which means $P(A \cap B) = P(A)$.

Bayes' theorem:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$$

Now we need to calculate $P(A)$ and $P(B)$.

For event A, the kth card is the largest value in the entire pack. All orderings of the cards are equally likely.

$$P(A) = \frac{1}{m}$$

For event B, the kth card is the largest among the first k cards. This means the largest card among the first k cards is in the kth position.

The probability of this is $\frac{1}{k}$ because the largest card among the first k cards is equally likely to be in any of the k positions.

Therefore:

$$P(A|B) = \frac{P(A)}{P(B)} = \frac{\frac{1}{m}}{\frac{1}{k}} = \frac{k}{m}$$

The probability that the kth card is the largest in the pack, given that it's the largest among the first k cards, is $\frac{k}{m}$.