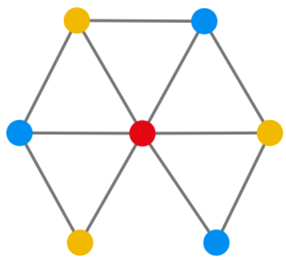
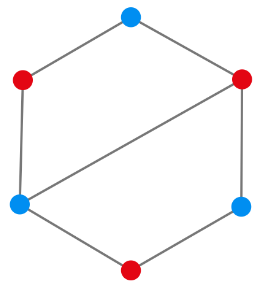
Recitation 7 solutions - CSCI 2200 (FOCS)

I.

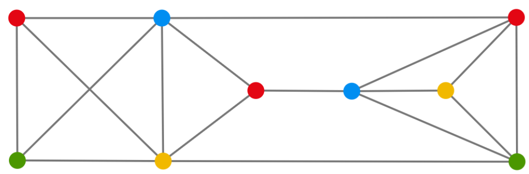
a) **3**



b) **2**

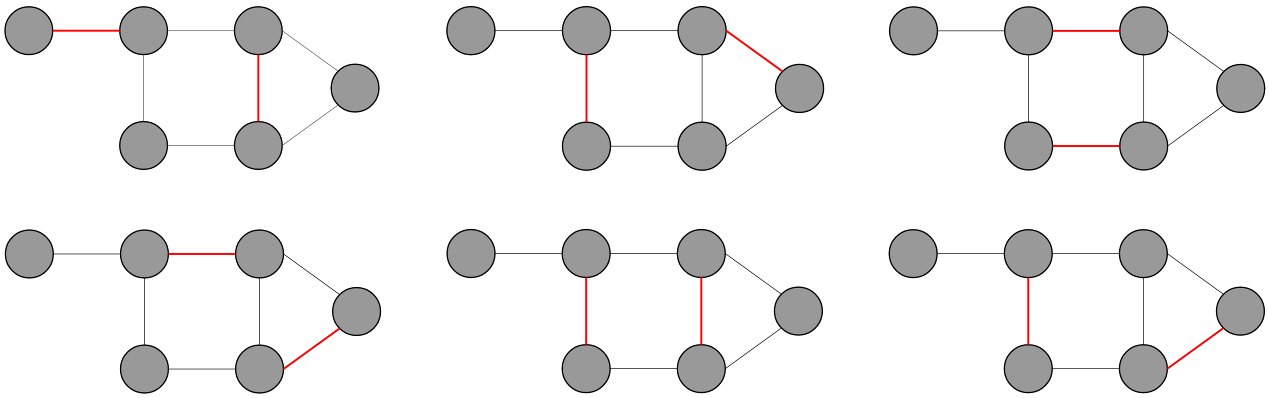


c) **4**

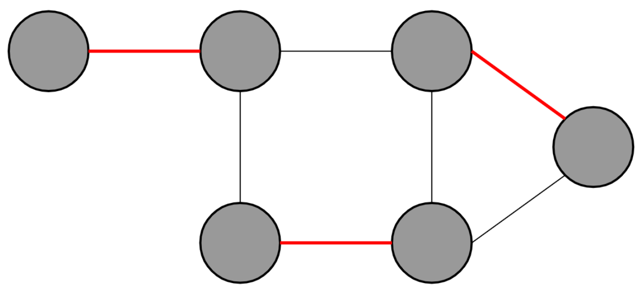


II.

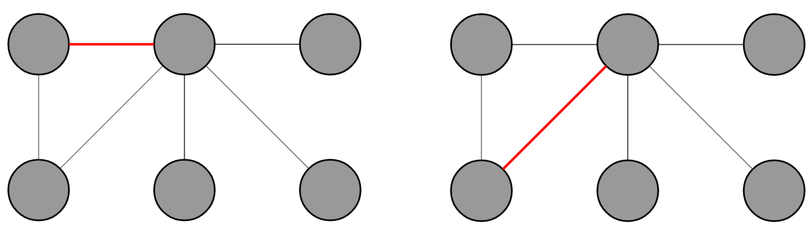
a) few variations:



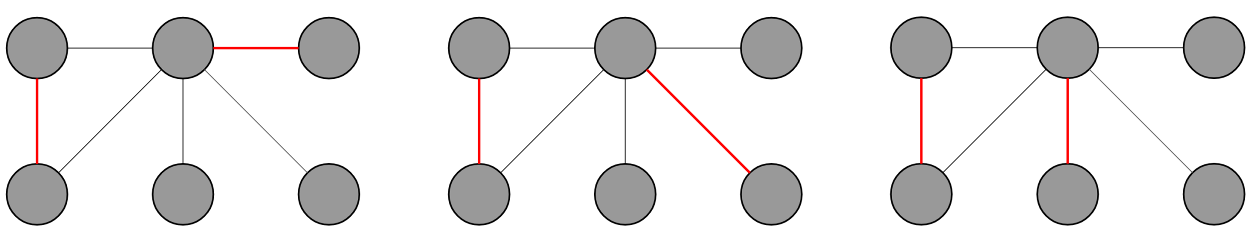
Maximum matching:



b) only two possibilities:



Maximum matching:



III.

Given:

- n kids want to go to summer camp
- There are m different summer camps
- Each kid k_i has a subset of camps they're willing to attend
- Each camp j has a limit l_j on the number of kids they can handle

Necessary and Sufficient Conditions:

For all kids to be able to attend a camp they want to go to, the following conditions must be met:

1. **Total Capacity Condition:**
The sum of all camp capacities must be greater than or equal to the total number of kids:

$$\sum_{j=1}^{m} l_j \geq n$$

2. **Hall's Theorem Extension:**
For any subset S of kids, let $C(S)$ be the set of all camps that at least one kid in S is willing to attend. Then:

$$\text{For all subsets } S \text{ of kids: } \sum_{j \in C(S)} l_j \geq |S|$$

This problem is an extension of the bipartite matching problem, where:

- One set of vertices represents the kids
- The other set represents the camps
- Edges exist between kids and the camps they're willing to attend
- Each camp vertex can match with multiple kids

The conditions work because:

1. **Total Capacity:**

- We need enough total spots to accommodate all kids

2. **Hall's Extension:**

- For any group of kids, the camps they're willing to attend must have enough total capacity

These conditions are:

- **Necessary:** If either condition fails, some subset of kids cannot be assigned to camps they're willing to attend
- **Sufficient:** If both conditions are met, there exists a valid assignment of kids to camps.

IV.

Given:

- 6 true/false questions
- 4 multiple choice questions with options A-E

Solution:
Let's use the Product Rule to solve this:
1. For each true/false question:

- 2 possible answers (true or false)
- This applies to all 6 T/F questions

2. For each multiple choice question:

- 5 possible answers (A through E)
- This applies to all 4 MC questions

3. Total number of possible answer sheets:

$$2^6 \cdot 5^4$$

$$\begin{aligned} 2^6 &= 64 \\ 5^4 &= 625 \\ 64 \cdot 625 &= 40{,}000 \end{aligned}$$

Therefore, there are 40,000 different possible answer sheets for this quiz.

V. Four students (Alex, Bailey, Chris, Dani) each get to pick a T-shirt. There are five different colors of T-shirt available.

1. **Same colors allowed:** Each of the 4 students can choose from any of the 5 colors independently, giving

   $$5^4 \;=\; 625$$

   possible ways.

2. **All different colors:** We must choose 4 distinct colors (out of 5) and assign them to the 4 students. The first student has 5 choices, the second then has 4 remaining choices, the third has 3, and the fourth has 2:

   $$5 \times 4 \times 3 \times 2 \;=\; 120.$$

VI. Prime factorization and number of divisors:
The integer 300056400 has the prime factorization

$$300056400 \;=\; 2^4 \cdot 3^7 \cdot 5^2 \cdot 7^3.$$

The number of positive divisors of a number $n = p_1^{a_1}\, p_2^{a_2}\, \cdots\, p_k^{a_k}$ is

$$(a_1 + 1)(a_2 + 1)\cdots(a_k + 1).$$

Because the combination of each prime factor to each possible power (including 0, hence the +1) gives all the divisors. Thus the number of divisors is

$$(4 + 1) \times (7 + 1) \times (2 + 1) \times (3 + 1) \;=\; 5 \times 8 \times 3 \times 4 \;=\; 480.$$

(VII) The most common Tarot deck has 78 cards.

(a) A simple type of Tarot reading involves drawing three cards: one for "past", one for "present", and one for "future". How many different drawings are possible?

Given the construction of our problem, the order in which we draw our cards matters. Since we are selecting cards without repleacement, we are looking for the number of 3-orderings of 78 objects which is given by the formula below:

$$\frac{78!}{(78-3)!} = 78 \cdot 77 \cdot 76 = 456,456$$

Therefore, there are $\boxed{456,456}$ possible drawings of three cards given that we put one card in past, present, and future.

(b) Tarot cards are also used as playing cards. One such game ("French Tarot") involves dealing six cards to a pool called le chien ("the dog"). How many sets of six cards are possible?

Since there is now no restriction on the order of these cards, counting the number of six card sets is equivalent to finding the number of 6-subsets of 78 objects which is given by the formula below:

$$\binom{78}{6} = \frac{78!}{(78-6)!6!} = 256,851,595$$

Therefore, there are $\boxed{256,851,595}$ possible sets of six tarot cards.

Recitation 7 Solutions

(VIII) To test if a graph with 50 vertices is 3-colorable, you decide to brute-force it and check all possible assignments of {red, green, blue} to vertices to see if any connected vertices share a color. Your computer can check 1,000,000 of these colorings every second. How long will it take to complete this process?

First we need to find the number possible vertex colorings. Since we have 50 vertices and 3 possible colors, the total number of colorings we have is 3^{50}.

Therefore,

$$3^{50} \text{ colorings} \cdot \frac{1 \text{ second}}{1,000,000 \text{ colorings}} \approx \boxed{7 \times 10^{17} \text{seconds}}$$

This turns out to be around 22.8 billion years of computation which is almost twice the age of the universe.