# Recitation 6 solutions - CSCI 2200 (FOCS)

Problem I and Problem II

I.

$$a|bc \ \wedge \ \gcd(a,b) = 1 \rightarrow a|c$$

*Direct Proof.*

Using Bezout's Identity, we know that:

$$ax + by = 1$$

Multiply both sides by c, then we get:

$$a(cx) + (bc)y = c$$

Since we can divide left had side, $(a|a(cx),\ a|(bc)y$, so, $a|(a(cx) + (bc)y))$, right hand side $c$, is also divisible by $a$. ■

II.

Calculate the number of zeros at the end of 1000!.

General idea is to find how many 10s in the factor of 1000!. Intuitively, we can think 10 as $2 \times 5$, and since 2 (as a factor) appears more rapidly than 5, we can count 5 instead.

Counting 5 as a factor is easy, since we can do $1000/5 = 200$, but there's other case, where 5 appears more than once in a number, such as $5^2$.

To address that, set up the formula,

$$\sum_{i=1}^{n} \left\lfloor \frac{1000}{5^i} \right\rfloor, \text{ where } n \in \mathbb{N},\ n = \max(k) \text{ that satisfies } (5^k \leq 1000)$$

Since $5^4 = 625$, $n = 4$. Solve the summation...

$$= \frac{1000}{5} + \frac{1000}{25} + \frac{1000}{125} + \frac{1000}{625} = 200 + 40 + 8 + 1(1.6, \text{ but round it down})$$
$$= 249$$

We have 259 0s at the end of 1000!.

## III. Prove that $\gcd(F_{n+1}, F_n) = 1$, where $F_n$ is the $n$th Fibonacci number

Prove using properties of GCD and Fibonacci numbers.

1. Let $d = \gcd(F_{n+1}, F_n)$. By definition of GCD:
   - $d$ divides $F_{n+1}$ ($d \mid F_{n+1}$)
   - $d$ divides $F_n$ ($d \mid F_n$)
2. By the definition of Fibonacci numbers:
   $$F_{n+1} = F_n + F_{n-1}$$
   Therefore:
   $$F_{n-1} = F_{n+1} - F_n$$
3. Since $d$ divides both $F_{n+1}$ and $F_n$, it must also divide their difference:
   $$d \mid (F_{n+1} - F_n)$$
   Therefore:
   $$d \mid F_{n-1}$$
4. We can continue this process:
   $$F_{n-1} = F_n - F_{n-1} \text{ so } d \mid F_{n-2}$$
   $$F_{n-2} = F_{n-1} - F_{n-2} \text{ so } d \mid F_{n-3}$$
   $$\vdots$$
   $$\text{Until we reach: } d \mid F_1$$
5. Since $F_1 = 1$, and $d \mid 1$, we must have $d = 1$
6. Therefore, $\gcd(F_{n+1}, F_n) = 1$ ■

# IV. Last digit of $3^{2025} + 4^{2025} + 7^{2025}$

**Solution.** We seek the last digit, i.e. the remainder when dividing by 10.

- Powers of 3 modulo 10 cycle with period 4:

  $$3^1 \equiv 3, \quad 3^2 \equiv 9, \quad 3^3 \equiv 7, \quad 3^4 \equiv 1 \pmod{10},$$

  and then repeat. Since $2025 \equiv 1 \pmod 4$, we get

  $$3^{2025} \equiv 3 \pmod{10}.$$

- Powers of 4 modulo 10 cycle with period 2:

  $$4^1 \equiv 4, \quad 4^2 \equiv 6 \pmod{10},$$

  and then repeat. Because 2025 is odd,

  $$4^{2025} \equiv 4 \pmod{10}.$$

- Powers of 7 modulo 10 also cycle with period 4:

  $$7^1 \equiv 7, \quad 7^2 \equiv 9, \quad 7^3 \equiv 3, \quad 7^4 \equiv 1 \pmod{10},$$

  and thus for $2025 \equiv 1 \pmod 4$,

  $$7^{2025} \equiv 7 \pmod{10}.$$

Hence,

$$3^{2025} + 4^{2025} + 7^{2025} \equiv 3 + 4 + 7 \equiv 14 \equiv 4 \pmod{10}.$$

**The last digit is 4.**

# VI. Proof that at least one of $G$ or $\overline{G}$ is connected

**Statement.** Let $G$ be any (undirected) graph on $n$ vertices, and let $\overline{G}$ denote its complement, i.e. the graph on the same vertices where two vertices are adjacent exactly when they are *not* adjacent in $G$. We claim that at least one of $G$ or $\overline{G}$ must be connected.

**Proof.** Suppose, for contradiction, that both $G$ and $\overline{G}$ are disconnected. Then $G$ can be partitioned into at least two nonempty connected components, say $C_1$ and $C_2$. By definition of disconnectedness in $G$, there are *no* edges between a vertex in $C_1$ and a vertex in $C_2$ in $G$. However, in the complement $\overline{G}$, precisely those missing edges are present; that is, every vertex in $C_1$ is adjacent to *every* vertex in $C_2$ in $\overline{G}$.

But if $\overline{G}$ were also disconnected, it would require that there is some way to separate its vertices into two groups with no edges between them in $\overline{G}$. Yet we have just seen that all cross-edges between $C_1$ and $C_2$ exist in $\overline{G}$. In fact, the presence of all edges between $C_1$ and $C_2$ in $\overline{G}$ ensures that any vertex in $C_1$ can reach any vertex in $C_2$ in a single step. This cannot happen in a disconnected graph. Consequently, $\overline{G}$ must be connected if $G$ is not.

Hence the assumption that both $G$ and $\overline{G}$ are disconnected is impossible. Therefore, at least one of the two graphs—$G$ or $\overline{G}$—is connected, as desired. □

## Problem V

(V) How many edges does each of these graphs have?

(a) $K_n$

This graph is a complete graph with $n$ vertices which means all $n$ vertices have degree $(n-1)$. Therefore,

$$\begin{aligned} |E| &= \frac{1}{2}\sum_{i=1}^{n} \delta_i \\ &= \frac{1}{2}\sum_{i=1}^{n}(n-1) \\ &= \boxed{\frac{n(n-1)}{2}} \end{aligned}$$

(b) $K_{m,n}$

This graph is a complete bipartide graph with $m$ vertices on one side and $n$ vertices on the other side. Therefore, the $m$ vertices all have degree $n$ and the $n$ vertices on the other side all have degree $n$.

$$\begin{aligned} |E| &= \frac{1}{2}\sum_{i=1}^{m+n} \delta_i \\ &= \frac{1}{2}\left(\sum_{i=1}^{m} n + \sum_{i=1}^{n} m\right) \\ &= \frac{2mn}{2} \\ &= \boxed{mn} \end{aligned}$$

(c) $W_n$

This graph is a wheel with $n-1$ outer vertices and one center vertex that connects directly to all other vertices. Therefore, we have $n-1$ outer vertices with degree 3 (two adjacent vertices and the center vertex) and 1 vertex with degree $(n-1)$.

$$\begin{aligned} |E| &= \frac{1}{2}\sum_{i=1}^{n} \delta_i \\ &= \frac{1}{2}\left(\sum_{i=1}^{n-1} 3 + (n-1)\right) \\ &= \frac{4(n-1)}{2} \\ &= \boxed{2(n-1)} \end{aligned}$$

# VII. A simple graph $G$ has $n$ vertices.

## (a) What is the minimum number of edges $G$ could have and still be connected?

Answer: $n - 1$ edges

If we were to build a connected graph from scratch:

- Start with one vertex
- To connect a new vertex to our graph, we need exactly one new edge
- For $n$ vertices, we need $n - 1$ edges to connect them all

This forms a tree. If we used fewer edges, some vertex would be isolated. A tree with $n$ vertices has exactly $n - 1$ edges. A tree is connected by definition.

## (b) What is the maximum number of edges $G$ could have and still not be connected?

Answer: $\frac{(n-1)(n-2)}{2}$ edges

To construct such a tree:

- Take one vertex and keep it completely isolated
- Take all remaining $n - 1$ vertices and connect them all to each other
- The $n - 1$ vertices can have $\frac{(n-1)(n-2)}{2}$ edges between them
- The formula $\frac{(n-1)(n-2)}{2}$ comes from the fact that with $n-1$ vertices, each vertex can connect to every other vertex except itself ($\binom{n-1}{2}$, the number of ways to choose 2 vertices from $n - 1$ vertices to form an edge). Try it out for yourself!

This is maximum because if we added any more edges, the isolated vertex would become connected to the rest of the graph, making the entire graph connected.

## Problem VIII

(VIII) **Claim** If every vertex in a simple graph $G$ has degree at least $d \geq 2$, then $G$ must contain a cycle of length at least $d+1$.

**Proof** Let $G$ be a graph with minimum degree $d \geq 2$, and let $P = (v_0, v_1, \ldots, v_{k-1}, v_k)$ be the longest simple path in $G$ (no vertices are repeated). Since each vertex in $G$ has degree at least 2 (because $d \geq 2$), the vertex $v_k$ must have at least 1 neighbor other than $v_{k-1}$.

Notice that all of the neighbors of $v_k$ must lie within $P$. This is because if $v_k$ had a neighbor that lies outside of $P$, we could extend $P$ by including that neighbor, which would contradict the assumption that $P$ is the longest simple path.

Let $v_i$ be the first vertex in $P$ that is a neighbor of $v_k$. Since $v_k$ has at least 1 neighbor other than $v_{k-1}$, we know that $v_i$ must appear before $v_{k-1}$ in the $P$.

Thus, we can form a cycle by taking the subpath from $v_i$ to $v_k$ and adding the edge $(v_i, v_k)$ to it. Since this cycle is simple, it length must be equal to the length of of the subpath used to generate it.

Since $v_i$ is the first vertex in $P$ that is a neighbor of $v_k$, the subpath from $v_i$ to $v_{k-1}$ must include all $d$ neighbors of $v_k$. Therefore, this subpath must have a length of at least $d$ which means that adding $v_k$ to this subpath gives us a minimum length of $d+1$. Therefore the cycle that we constructed earlier must have a length of at least $d+1$.

Therefore, we have shown that if every vertex in a simple graph $G$ has degree at least $d \geq 2$, then $G$ must contain a cycle of length at least $d+1$. ■