Problem I

(I) **Claim** For all $n \in \mathbb{N}_0$, the number of palindromes of length n is $2^{\lceil n/2 \rceil}$.

Proof We will prove that the number of palindromes of length n is $2^{\lceil n/2 \rceil}$ for all $n \in \mathbb{N}_0$ by structural induction.

Recall the recursive definition of the set of all palindromes $\mathcal{P}$ as given below:

(1) $\varepsilon, 0, 1 \in \mathcal{P}$

(2) $x \in \mathcal{P} \implies (0x0 \in \mathcal{P} \wedge 1x1 \in \mathcal{P})$

We will take is as a given for this question that $\mathcal{P}$ contains all binary palindromes and that each palindrome in $\mathcal{P}$ has a unique generation.

[**Base Cases**] When $n = 0$, there is only one possible palindrome (ε), which is equal to $2^{\lceil 0/2 \rceil}$.
When $n = 1$, there are only two possible palindromes (0 and 1), which is equal to $2^{\lceil 1/2 \rceil}$.

[**Induction Step**] Suppose that for some $n \in \mathbb{N}_0$ that the number of palindromes of length n is $2^{\lceil n/2 \rceil}$. We wish to show that that the number of palindromes of length $n+2$ is $2^{\lceil (n+2)/2 \rceil}$.

Assuming that each palindrome has a unique generation by $\mathcal{P}$, we know that each palindrome of length n will generate two palindromes of length $n+2$. Therefore,

$$\begin{aligned}
\#\text{ of palindromes of length } (n+2) &= 2 \cdot \#\text{ of palindromes of length } (n) \\
&= 2 \cdot 2^{\lceil n/2 \rceil} && \text{by our Induction Hypothesis} \\
&= 2^{\lceil n/2 \rceil + 1} \\
&= 2^{\lceil (n+2)/2 \rceil}
\end{aligned}$$

Therefore, if the number of palindromes of length n is $2^{\lceil n/2 \rceil}$ for some $n \in \mathbb{N}_0$ then the number of palindromes of length $n+2$ is $2^{\lceil (n+2)/2 \rceil}$.

Thus we have shown by structural induction that for all $n \in \mathbb{N}_0$, the number of palindromes of length n is $2^{\lceil n/2 \rceil}$. ■

Problem II

II. Matrix multiplication involves multiplying rows of the first matrix by columns of the second matrix. The values in the product matrix are the sums of these products. In the particular case of multiplying 2x2 matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Prove that $\forall n \in \mathbb{N}$, $A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$, where F_n is the nth Fibonacci number (assuming $F_0 = 0$ and $F_1 = 1$).

Proof (by induction).

Base case ($n = 1$)*:*

$$A^1 = A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $F_2 = 1$, $F_1 = 1$, and $F_0 = 0$, we have

$$\begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

so the formula holds for $n = 1$.

Inductive step: Assume for some $n \geq 1$ that

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

We must show

$$A^{n+1} = \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix}.$$

$$A^{n+1} = A^n A = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} + F_n & F_{n+1} \\ F_n + F_{n-1} & F_n \end{pmatrix}.$$

Using the Fibonacci recurrence relation $F_{k+2} = F_{k+1} + F_k$, we recognize

$$F_{n+1} + F_n = F_{n+2} \quad \text{and} \quad F_n + F_{n-1} = F_{n+1}.$$

Thus,

$$A^{n+1} = \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix},$$

which matches the required form with n replaced by $n + 1$.

By the principle of mathematical induction, the claim holds for all $n \geq 1$.

Problem III

III. Find closed-form expressions for the following sums:

(a) $\sum_{i=1}^{n}(3i+2i^2)$

$$\begin{aligned}
\sum_{i=1}^{n}(3i+2i^2) &= 3\sum_{i=1}^{n} i + 2\sum_{i=1}^{n} i^2 \quad \text{(split into two sums)} \\
&= 3\cdot\frac{n(n+1)}{2} + 2\cdot\frac{n(n+1)(2n+1)}{6} \quad \sum i = \frac{n(n+1)}{2} \text{ and } \sum i^2 = \frac{n(n+1)(2n+1)}{6}) \\
&= \frac{9n(n+1)}{6} + \frac{2n(n+1)(2n+1)}{6} \quad \text{(finding common denominator of 6)} \\
&= \frac{n(n+1)(9+4n+2)}{6} \quad \text{(simplify)} \\
&= \frac{n(n+1)(4n+11)}{6}
\end{aligned}$$

(b) $\sum_{i=1}^{n}(-1)^i i$

Let's examine the pattern and split into even and odd terms:

$$\begin{aligned}
\sum_{i=1}^{n}(-1)^i i &= -1+2-3+4-...+(-1)^n n \\
&= (2+4+...+2\lfloor\frac{n}{2}\rfloor) - (1+3+...+(2\lceil\frac{n}{2}\rceil-1)) \quad \text{(splitting into even and odd terms)} \\
&= \sum_{i=1}^{\lfloor n/2\rfloor} 2i - \sum_{i=1}^{\lceil n/2\rceil}(2i-1) \quad \text{(rewriting as sums)} \\
&= 2\sum_{i=1}^{\lfloor n/2\rfloor} i - \sum_{i=1}^{\lceil n/2\rceil}(2i-1) \quad \text{(factoring out 2 from first sum)} \\
&= 2\cdot\frac{\lfloor n/2\rfloor(\lfloor n/2\rfloor+1)}{2} - \sum_{i=1}^{\lceil n/2\rceil}(2i-1) \quad \text{(using sum formula } \sum_{i=1}^{k} i = \frac{k(k+1)}{2}) \\
&= \lfloor n/2\rfloor(\lfloor n/2\rfloor+1) - (2\sum_{i=1}^{\lceil n/2\rceil} i - \sum_{i=1}^{\lceil n/2\rceil} 1) \quad \text{(split the second sum)} \\
&= \lfloor n/2\rfloor(\lfloor n/2\rfloor+1) - (2\cdot\frac{\lceil n/2\rceil(\lceil n/2\rceil+1)}{2} - \lceil n/2\rceil) \quad \text{(using sum formulas)} \\
&= \lfloor n/2\rfloor(\lfloor n/2\rfloor+1) - (\lceil n/2\rceil(\lceil n/2\rceil+1) - \lceil n/2\rceil) \\
&= \lfloor n/2\rfloor(\lfloor n/2\rfloor+1) - \lceil n/2\rceil^2
\end{aligned}$$

Simplify this expression based on whether n is even or odd:
When n is even:

- $\lfloor n/2 \rfloor = n/2$ and $\lceil n/2 \rceil = n/2$
- Expression becomes $(n/2)(n/2+1) - (n/2)^2 = n/2$
- Note that when n is even, $(-1)^n = 1$, giving us $n/2$

When n is odd:

- $\lfloor n/2 \rfloor = (n-1)/2$ and $\lceil n/2 \rceil = (n+1)/2$
- Expression becomes $\frac{n-1}{2}(\frac{n+1}{2}) - (\frac{n+1}{2})^2 = -\frac{n+1}{2}$
- Note that when n is odd, $(-1)^n = -1$, giving us $-\frac{n+1}{2}$

Therefore:

$$= (-1)^n \lceil n/2 \rceil$$

(c) $\sum_{i=1}^{n} \sum_{j=1}^{m} (i+j)$

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j=1}^{m} (i+j) &= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} (i+j) \right) \quad \text{(starting with inner sum)} \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} i + \sum_{j=1}^{m} j \right) \quad \text{(splitting sum of the sum)} \\
&= \sum_{i=1}^{n} \left(m \cdot i + \sum_{j=1}^{m} j \right) \\
&= \sum_{i=1}^{n} \left(mi + \frac{m(m+1)}{2} \right) \quad \text{(using sum formula } \sum_{j=1}^{m} j = \frac{m(m+1)}{2}\text{)} \\
&= m \sum_{i=1}^{n} i + \sum_{i=1}^{n} \frac{m(m+1)}{2} \quad \text{(distributing the sum)} \\
&= m \sum_{i=1}^{n} i + n \cdot \frac{m(m+1)}{2} \\
&= m \cdot \frac{n(n+1)}{2} + \frac{nm(m+1)}{2} \quad \text{(using sum formula } \sum_{i=1}^{n} i = \frac{n(n+1)}{2}\text{)} \\
&= \frac{mn(n+1)}{2} + \frac{nm(m+1)}{2} \quad \text{(writing terms with similar denominators)} \\
&= \frac{nm(n+1) + nm(m+1)}{2} \quad \text{(simplify)} \\
&= \frac{nm(n+1+m+1)}{2} \quad \text{(simplify)} \\
&= \frac{nm(n+m+2)}{2}
\end{aligned}$$

IV.

In increasing order:

$$n \rightarrow (\log n)^3 \rightarrow n \log n \rightarrow n\sqrt{n} \rightarrow 2^n \rightarrow n! \rightarrow n^n \rightarrow n^{100}$$

V.

(a),

$$T_n = T_{n-1} + n^2$$
$$T_n = T_{n-2} + (n-1)^2 + n^2$$
$$T_n = T_{n-3} + (n-2)^2 + (n-1)^2 + n^2$$

Thus:

$$T_n = T_0 + \sum_{i=1}^{n} i^2 = T_0 + \frac{n(n+1)(2n+1)}{6} \approx \Theta(n^3)$$

(b),

$$T_n = 2T_{n-1} - T_{n-2} + 2$$

input numbers:

0	1	2	3	4	5	6
1	2	5	10	17	26	37
-1						
0	1	4	9	16	25	36

Real formula:

$$T_n = n^2 + 1 \approx \Theta(n^2)$$