Recitation 4 solutions - CSCI 2200 (FOCS)

Problem I

I. Prove the following using basic induction:

$\forall n \geq 2$ ($n \in \mathbb{N}$), a $2^n \times 2^n$ square grid with a 2×2 corner removed can be precisely tiled using L-shaped tiles of size 3. ("Precisely tiled" means covering the full area with no tiles overlapping or hanging off the edge of the board. Tiles may be rotated or flipped. A sample L-tile, board with n=2, and board with n=3 are show.)

We will prove this by induction.

Base case: $n = 2$

(1) For $n = 2$, we have a 4×4 grid with a 2×2 corner removed

(2) This leaves 12 squares to be covered.

(3) As shown in the example diagram A, this can be covered exactly by 4 L-shaped tiles

Inductive step: Assume that for some $k \geq 2$, a $2^k \times 2^k$ grid with a 2×2 corner removed can be tiled with L-shaped pieces. We will prove this holds for $n = k + 1$. Refer to example diagram B.

(1) For $n = k + 1$, we have a $2^{k+1} \times 2^{k+1}$ grid with a 2×2 corner removed

(2) Divide this grid into four quadrants, each of size $2^k \times 2^k$

(3) The 2×2 removed corner lies entirely within one quadrant

(4) For the quadrant containing the removed corner:

- By our inductive hypothesis, this can be tiled with L-shaped pieces

(5) Where the other three quadrants meet, place one L-shaped tile to connect them

(6) The remaining areas in each of these three quadrants are now $2^k \times 2^k$ grids with 2×2 corners removed

(7) By our inductive hypothesis, each of these can be tiled with L-shaped pieces

Since we've shown both the base case and inductive step, we have proved by induction that a $2^n \times 2^n$ square grid with a 2×2 corner removed can be precisely tiled using L-shaped tiles for all $n \geq 2$. ■

Diagram A

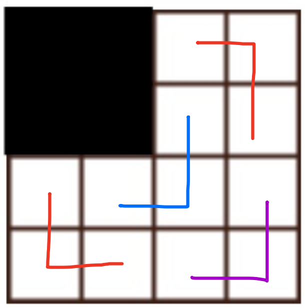
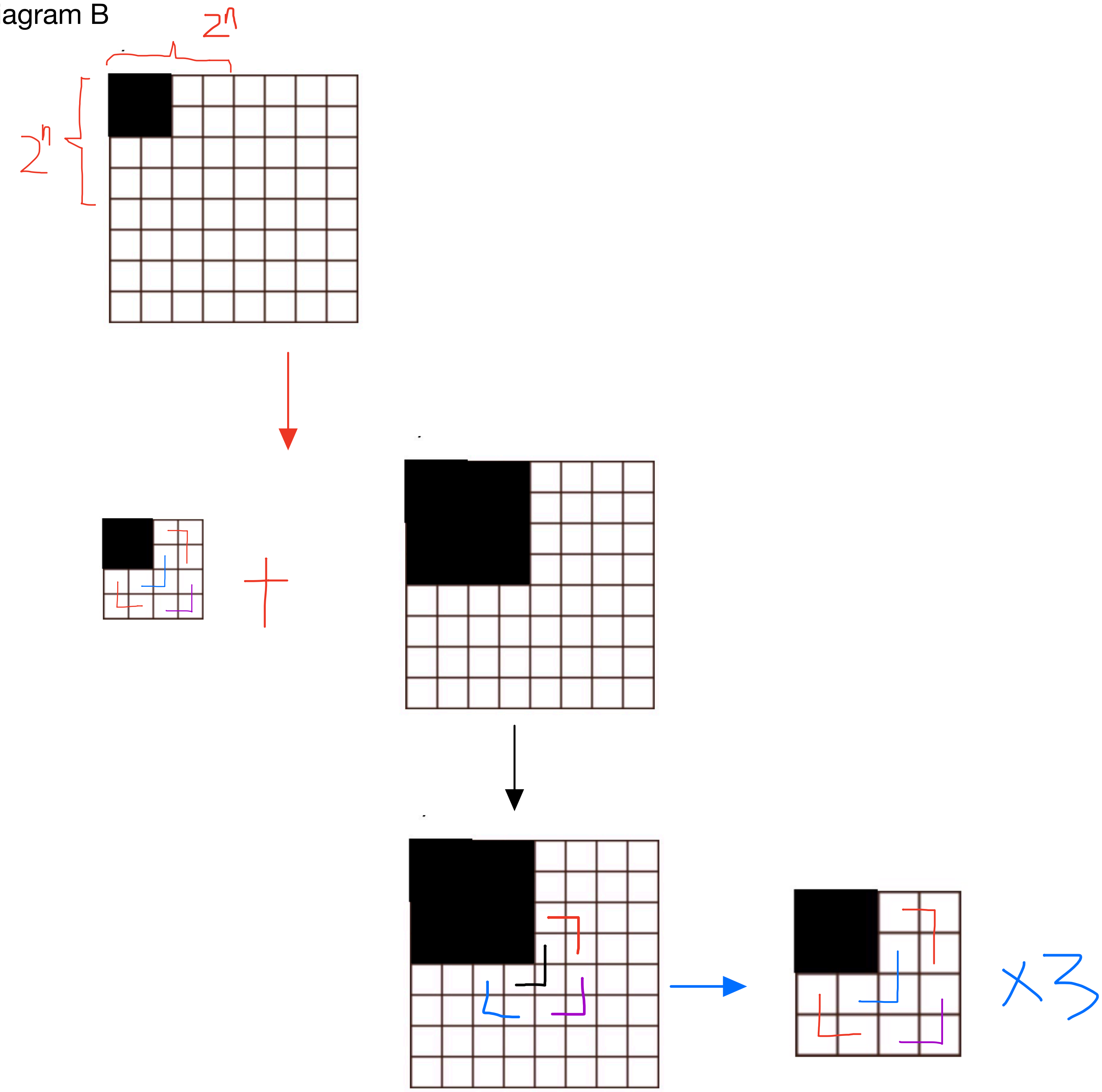


Diagram B



Problem II and Problem V

II. Prove using Leaping Induction: for all $n \in \mathbb{N}_0$, the expression $n^2 + n + 1$ is not divisible by 5.
Using step size of 5:

Base Cases:

$$\begin{aligned}
n = 0, &\quad 0^2 + 0 + 1 = 1, \quad \text{not divisible by 5} \\
n = 1, &\quad 1^2 + 1 + 1 = 3, \quad \text{not divisible by 5} \\
n = 2, &\quad 2^2 + 2 + 1 = 7, \quad \text{not divisible by 5} \\
n = 3, &\quad 3^2 + 3 + 1 = 13, \quad \text{not divisible by 5} \\
n = 4, &\quad 4^2 + 4 + 1 = 21, \quad \text{not divisible by 5}
\end{aligned}$$

Inductive Step:
Assume that for $n = k \in \mathbb{N}_0$, the expression $k^2 + k + 1$ is not divisible by 5. for $n = k + 5$:

$$\begin{aligned}
&(k + 5)^2 + (k + 5) + 1 \\
&= k^2 + 10k + 25 + k + 5 + 1 \\
&= (k^2 + k + 1) + 10k + 25 + 5 \\
&= (k^2 + k + 1) + 5(2k + 5 + 1)
\end{aligned}$$

Since $k^2 + k + 1$ is not divisible by 5, $= (k^2 + k + 1) + 5(2k + 5 + 1)$ is also not divisible by 5.

This proves that if it holds for $n = k$, it holds for $n = k + 5$. Thus by leap induction, we proved that for all $n \in \mathbb{N}_0$, the expression n^2+n+1 is not divisible by 5.

V. Recursive Definition of $\{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, ...\}$:

$$f(n) = \begin{cases} 1 & n = 1 \\ 2 & n = 2 \\ 3 & n = 3 \\ 4 & n = 4 \\ f(n-4) + 5 & n > 4 \end{cases}$$

Problem III

(III) **Claim** In any directed graph in which every pair of vertices has a single one-way link between them, there is some route that touches all vertices. Prove this claim using strong induction.

Proof We will prove that for all $n \in \mathbb{N}$ that a directed graph with n vertices in which every pair of vertices has a single one-way link between them, there is some path that touches all vertices.

[**Base Case**] A graph with one vertex clearly has a path that touches all the vertices (just the vertex itself).

[**Induction Step**] Suppose that for all $1 \leq n \leq k$, any directed graph with n vertices in which every pair of vertices has a single one-way link between them has path that touches all vertices. We will show that a directed graph with $k+1$ vertices in which every pair of vertices has a single one-way link between them has a path that touches all vertices.

Let G be a directed graph that satisfies the critera given above where $|V| = k+1$ and let v be an arbitrary vertex of G. By the construction of G, we know that for every other vertex u in G, there exists exactly one directed edge that connects u and v. Therefore, we can partition the set of vertices excluding v into two sets V_{in} and V_{out} where V_{in} contains all vertices u such that the edge (u, v) is in G and V_{out} contains all vertices u such that the edge (v, u) is in G.

Let G_{in} and G_{out} be the subgraphs of G induced by V_{in} and V_{out} respectively.
By construction of G, both of these subgraphs must both satisfy the property that every pair of vertices has a single one-way link between them.
Since $|V| = |V_{\text{in}}| + |V_{\text{out}}| + 1$, it follows that if one of these partitions is non-empty, then the corresponding induced subgraph will have between 1 and k vertices which means that by our induction hypothesis that there exists a path that touches all vertices.
In addition, since G must have at least 2 vertices, at least one of these graphs is guaranteed to be non-empty. Therefore we can consider three cases:

(1) $V_{\text{in}} = \emptyset$ and $V_{\text{out}} \neq \emptyset$

(2) $V_{\text{in}} \neq \emptyset$ and $V_{\text{out}} = \emptyset$

(3) $V_{\text{in}} \neq \emptyset$ and $V_{\text{out}} \neq \emptyset$

In case (1), we know that for every vertex u not equal to v, there exists an edge (v, u). Since G_{out} is non-empty, it has a path that touches all vertices and thus there exists a path in G starting at v that connects to the start of the path in G_{out}.

In case (2) we know that for every vertex u not equal to v, there exists an edge (u, v). Since G_{in} is non-empty, it has a path that touches all vertices and thus there exists a path in G which is precisely the path in G_{in} with v appended.

In case (3), we know that both G_{in} and G_{out} are both non-empty, both graphs must have paths P_{in} and P_{out} such that these paths touch all vertices in their respective graphs. Therefore we can construct the path $P_{\text{in}} \to v \to P_{\text{out}}$ which will touch all vertices in G.

Therefore, we have shown that if any directed graph with n vertices in which every pair of vertices has a single one-way link between them has path that touches all vertices for all $1 \leq n \leq k$ then a graph with the same construction with $k+1$ vertices has a path that touches all vertices.

Thus we have proven our claim by strong induction. ■

Problem IV

IV.

a)

$$T_0 = 2; T_n = T_{n-1} + 3n \text{ for } n \geq 1$$

Create a table to find out the difference:

#	0	1	2	3	4
0th	2	5	11	20	32
1st	-	3	6	9	12

for each n, $3n$ is added to final value. So, we can set up summation rule for 3,

$$\sum_{i=0}^{n} 3n = 3 \times \sum_{i=0}^{n} n = 3 \times \frac{n(n+1)}{2}$$

This will make the final values:

#	0	1	2	3	4
-	0	3	9	18	30

So simply add 2 at the end, so the final formular will be:

$$T_n = \frac{3n(n+1)}{2} + 2$$

b)

$$T_0 = 3; T_n = 2 \times T_{n-1} + n \text{ for } n \geq 1$$

Similar approach:

#	0	1	2	3	4	5	6
0th	3	7	16	35	74	153	312
1st	-	4	9	19	39	79	159
2nd	-	-	5	10	20	40	80

From here, we can find out that for each n, the value grows exponentially, or, by $5 \times 2^{n-2}$ at 2nd order difference row.

Then, to calculate 1st order row...

$$\sum_{i=0}^{n-2} 5 \times 2^i = 5 \times \sum_{i=0}^{n-2} 2^i = 5 \times (2^{n-1} - 1)$$

...and +4 as a constant.

Here, since we conclude that for every n, $5(2^{n-1} - 1) + 4$ is added to the final value, we can set up:

$$\sum_{i=0}^{n} 5(2^{i-1} - 1) + 4 = 4n + 5 \times \left(\sum_{i=0}^{n} 2^{i-1} - 1 \right)$$

$$= 4n - 5n + 5 \times \sum_{i=0}^{n} 2^{i-1} = 4n - 5n + 5 \times (2^n - 1) = 5(2^n - 1) - n$$

The final formula makes table:

#	0	1	2	3	4
-	0	4	13	32	71

...so don't forget to add constant 3.

$$T_n = 5(2^n - 1) - n + 3, \text{ or, } = 5 \times 2^n - n - 2$$

Problem VI

(VI) What is contained in the following recursively defined set $\mathcal{S}$?

(1) $3, 4 \in \mathcal{S}$

(2) $x, y \in \mathcal{S} \Rightarrow x + y \in \mathcal{S}$

Assuming minimality, we know that 3 and 4 are both in $\mathcal{S}$. We can clearly generate the following values using our constructor rule:

$$3 + 3 = 6 \in \mathcal{S}, \quad 3 + 4 = 7 \in \mathcal{S}, \quad 4 + 4 = 8 \in \mathcal{S}$$

Therefore, since we have three consecutive values in $\mathcal{S}$, we can generate all natural numbers greater than or equal to 6. A good guess for what our set $\mathcal{S}$ should be is $\mathbb{N} \setminus \{1, 2, 5\}$ which is all the natural numbers excluding 1, 2, and 5. Here is a proof (not required for this specific problem) that these two sets are equal.

Claim $\mathcal{S} = \mathbb{N} \setminus \{1, 2, 5\}$.

Proof To prove that these sets are equal, we must show that $\mathcal{S} \subseteq \mathbb{N} \setminus \{1, 2, 5\}$ and that $\mathbb{N} \setminus \{1, 2, 5\} \subseteq \mathcal{S}$.

We will first show that $\mathcal{S} \subseteq \mathbb{N} \setminus \{1, 2, 5\}$ by showing that $\mathcal{S} \subseteq \mathbb{N}$ and that $\{1, 2, 5\} \cap \mathcal{S} = \emptyset$.
To show that $\mathcal{S} \subseteq \mathbb{N}$, we will use structural induction using the claim if $x \in \mathcal{S}$, $x \in \mathbb{N}$.

[**Base Case**] It is clear that both $3, 4 \in \mathbb{N}$.

[**Induction Step**] Suppose that for some $x, y \in \mathcal{S}$ that $x, y \in \mathbb{N}$. Since $\mathbb{N}$ is closed under addition, we know that $x + y \in \mathbb{N}$.

Therefore, $\mathcal{S} \subseteq \mathbb{N}$ by structural induction. We will now show that $\mathcal{S} \cap \{1, 2, 5\} = \emptyset$ by contradiction.

Suppose for the sake of contradiction that $\mathcal{S} \cap \{1, 2, 5\} \neq \emptyset$. Therefore, there exists some $n \in \{1, 2, 5\}$ such that $n \in \mathcal{S}$. Since we have already shown that $\mathcal{S} \subseteq \mathbb{N}$, we know that $n = x + y$ where $x, y \in \mathbb{N}$. Using this knowledge, it follows that $n \geq 2$ which means that $n \neq 1$. That then tells us that $x, y \geq 2$ which means that $n \geq 4$. Therefore we know that 3 must be the minumum element of $\mathcal{S}$ and thus $n \geq 6$. This however is a contradiction since we assumed that $n \in \{1, 2, 5\}$ which means that $n < 6$.

Therefore, we have shown that $\mathcal{S} \cap \{1, 2, 5\} = \emptyset$ by contradiction which tells us that $\mathcal{S} \subseteq \mathbb{N} \setminus \{1, 2, 5\}$.

Next, we will show that $\mathbb{N} \setminus \{1, 2, 5\} \subseteq \mathcal{S}$. Let $n \in \mathbb{N} \setminus \{1, 2, 5\}$ We consider two cases: $n \in \{3, 4\}$ and $n \geq 6$. If $n \in \{3, 4\}$ then it is clear by our basis that $n \in \mathcal{S}$. Otherwise, if $n \geq 6$, we will show that $n \in \mathcal{S}$ by 3-leaping induction.

[**Base Cases**] We will use three base cases of 6, 7, 8. We can represent $6 = 3 + 3$, $7 = 4 + 3$, and $8 = 4 + 4$ which means that by our constructor, $6, 7, 8 \in \mathcal{S}$.

[**Induction Step**] Suppose that for some $n \geq 6$, $n \in \mathcal{S}$. Therefore, since $3 \in \mathcal{S}$, it follows that $n + 3 \in \mathcal{S}$ by our constructor.

Therefore we have shown that if $n \geq 6$ then $n \in \mathcal{S}$ by 3-leaping induction. This means that $\mathbb{N} \setminus \{1, 2, 5\} \subseteq \mathcal{S}$.

Since we have shown that $\mathcal{S} \subseteq \mathbb{N} \setminus \{1, 2, 5\}$ and $\mathbb{N} \setminus \{1, 2, 5\} \subseteq \mathcal{S}$, we have proven our claim that $\mathcal{S} = \mathbb{N} \setminus \{1, 2, 5\}$. ■