Recitation 3 solutions - CSCI 2200 (FOCS)

Problem I

I. Prove using a direct proof: $\forall n \in \mathbb{N}$, if n is not divisible by 3, then $n^2 \div 3$ has a remainder of 1. (Hint: Two cases.)

Since n is not divisible by 3, n must have remainder 1 or 2 when divided by 3. We consider these two cases:

- Case 1: n has remainder 1 when divided by 3
  - Then $n = 3k + 1$ for some $k \in \mathbb{N}$
  - $n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$
  - Therefore, $n^2 \div 3$ has remainder 1
- Case 2: n has remainder 2 when divided by 3
  - Then $n = 3k + 2$ for some $k \in \mathbb{N}$
  - $n^2 = (3k+2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$
  - Therefore, $n^2 \div 3$ has remainder 1
- In both cases, $n^2 \div 3$ has remainder 1. Thus, we have proved our claim. ■

Problem II and Problem VI

(II) Prove using contraposition: If x and y are positive integers such that $xy < 10000$, then $x < 100 \vee y < 100$.

Proof Let x and y be positive integers such that $x \geq 100$ and $y \geq 100$. Therefore, $xy > 10000$. Since we have proven the contrapositive of the original claim, we have proven the original claim. ■

(VI) Prove the following by induction:

$$\forall n \in \mathbb{N}, 1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}$$

Proof Let $P(n)$ be given by the claim $\sum_{i=1}^{n}(3i - 2) = \frac{n(3n-1)}{2}$.
We will prove that $P(n)$ is true for all $n \in \mathbb{N}$ by induction.

First we will show that $P(1)$ is true. Manipulating the left-hand side of $P(1)$ we have that

$$\begin{aligned}\sum_{i=1}^{1}(3i-2) &= 3 - 2 \\ &= 1 \\ &= \frac{1 \cdot (3(1) - 1)}{2}\end{aligned}$$

which is equal to the right-hand side of $P(1)$. Therefore, $P(1)$ is true.

Next, we will assume that $P(k)$ is true for some $k \in \mathbb{N}$ and show that $P(k+1)$ is true.
Using the left-hand side of $P(k+1)$ we have

$$\begin{aligned}\sum_{i=1}^{k+1}(3i-2) &= \sum_{i=1}^{k}(3i-2) + (3(k+1) - 2) \\ &= \frac{k(3k-1)}{2} + (3(k+1) - 2) && \text{by our induction hypothesis} \\ &= \frac{3k^2 - k}{2} + 3k + 1 \\ &= \frac{3k^2 - k + 6k + 2}{2} \\ &= \frac{3k^2 + 5k + 2}{2} \\ &= \frac{(k+1)(3k+2)}{2} \\ &= \frac{(k+1)(3(k+1) - 1)}{2}\end{aligned}$$

Since we have shown that the left and right-hand sides of $P(k+1)$ are equal, it follows that if $P(k)$ is true for some $k \in \mathbb{N}$ then $P(k+1)$ is true.

Thus we have shown that for all $n \in \mathbb{N}$, $\sum_{i=1}^{k+1}(3i-2) = \frac{n(3n-1)}{2}$ by induction. ■

Problem III

3.

Given. $a^2 - b^2 = 6$
Condition a, b are positive integer.

proof by contradiction. Assume that given conditions, there exists a and b that satisfies the equation. Then,

$$a^2 - b^2 = (a+b)(a-b) = 6$$

Since a, b are positive integer, $(a+b), (a-b)$ are both positive, with $(a+b) > (a-b)$
Factorization of 6 is 1*6 or 2*3. Given the condition we just derived, we have two cases:

$$(a+b) = 3, (a-b) = 2$$
$$\text{or}$$
$$(a+b) = 6, (a-b) = 1$$

However, for each case, a will be $a = 2.5$, $a = 3.5$ respectively, which contradicts with our first assumption, that a and b are both integers.
Therefore, by contradiction, there are no positive integers a and b such that $a^2 - b^2 = 6$. ■

IV. (a) *Disprove: If $\sqrt{r}$ is irrational, then r is irrational.*

Counterexample. Take $r = 2$. We know $\sqrt{2}$ is irrational, yet 2 itself is a rational number. Hence one has $\sqrt{r}$ irrational but r rational, so the statement is false.

IV. (b) *Disprove: $\forall n \in \mathbb{N}$, $3^n + 2$ is prime.*

Counterexample. Take $n = 5$. Then

$$3^5 + 2 = 243 + 2 = 245 = 5 \times 49,$$

which is not prime. Thus the statement is false.

VIII. *Prove using induction that $\forall n \in \mathbb{N}$,*

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

Base Case: Let $n = 1$. Then

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2}.$$

$$\frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}.$$

Hence, the statement holds for $n = 1$.

Inductive Step: Assume the statement holds for some $n = k \in \mathbb{N}$, i.e.,

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}.$$

We need to show it holds for $n = k + 1$. Consider

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{1}{(k+1)(k+2)} \sum_{i=1}^{k} \frac{1}{i(i+1)}.$$

By the induction hypothesis, we get

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{1}{(k+1)(k+2)} + \frac{k}{k+1}.$$

Find a common denominator:

$$\frac{k}{k+1} = \frac{k(k+2)}{(k+1)(k+2)},$$

so

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{1}{(k+1)(k+2)} + \frac{k(k+2)}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)}.$$

Observe that

$$k(k+2)+1 = k^2+2k+1 = (k+1)^2,$$

thus

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} = \frac{(k+1)}{(k+1)+1}.$$

This matches the form $\frac{n}{n+1}$ when $n = k+1$. Hence by the principle of induction, $\forall n \in \mathbb{N}, \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

Problem V

Prove the following by induction:
V. $\forall n \in \mathbb{N}$, $9^n + 3$ is divisible by 12.

Base case: $n = 1$

(1) When $n = 1$, we have $9^1 + 3 = 12$

(2) Since 12 is divisible by 12, the claim holds for $n = 1$

Inductive step: Assume that $9^k + 3$ is divisible by 12 for some $k \geq 1$. We want to show that $9^{k+1} + 3$ is also divisible by 12.

(1) Since we assumed $9^k + 3$ is divisible by 12, we can write $9^k + 3 = 12m$ for some integer m

(2) Therefore, $9^k = 12m - 3$

(3) For $9^{k+1} + 3$:

$$\begin{aligned} 9^{k+1} + 3 &= 9 \cdot 9^k + 3 \\ &= 9(12m - 3) + 3 \text{ (substituting what we know about } 9^k) \\ &= 108m - 27 + 3 \\ &= 108m - 24 \\ &= 12(9m - 2) \end{aligned}$$

(4) Since $(9m - 2)$ is an integer, $9^{k+1} + 3$ is divisible by 12

(5) We have shown that if $9^k + 3$ is divisible by 12 for some $k \geq 1$, $9^{k+1} + 3$ is also divisible by 12.

Since we have shown both the base case and inductive step, we have proved that $\forall n \in \mathbb{N}$, $9^n + 3$ is divisible by 12 via induction. ■

7.

$$\forall n \in \mathbb{N},\ 2^{n+2} + 3^{2n+1} \text{ is divisible by } 7.$$

proof by induction.

Base case $n = 1$,

$$2^{1+2} + 3^{2+1} = 8 + 27 = 35 = 7 * 5 \text{ (True)}$$

Inductive Step,
Assume that:

$$2^{n+2} + 3^{2n+1} \text{ is divisible by } 7,$$
$$\text{then}$$
$$2^{(n+1)+2} + 3^{2(n+1)+1} \text{ is also divisible by } 7$$

Or,

$$2^{n+2} + 3^{2n+1} \text{ is divisible by } 7 \rightarrow 2^{(n+1)+2} + 3^{2(n+1)+1} \text{ is also divisible by } 7$$

From right hand side, modify equation:

$$\begin{aligned} 2^{(n+1)+2} + 3^{2(n+1)+1} &= 2^{n+3} + 3^{2n+3} \\ &= 2 \times 2^{n+2} + 9 \times 3^{2n+1} \\ &= 2 \times 2^{n+2} + 2 \times 3^{2n+1} + 7 \times 3^{2n+1} \\ &= 2 \times (2^{n+2} + 3^{2n+1}) + 7 \times 3^{2n+1} \end{aligned}$$

Here, we already assumed that $2^{n+2} + 3^{2n+1}$ is divisible by 7.

$$\begin{aligned} &= 2 \times (7k) + 7 \times 3^{2n+1} \\ &= 7 \times (2k + 3^{2n+1}) \end{aligned}$$

Therefore, we shown that the right hand side equation is divisible by 7.
Thus, by induction,

$$\forall n \in \mathbb{N},\ 2^{n+2} + 3^{2n+1} \text{ is divisible by } 7$$

is true. ■