Recitation 2 Solutions - CSCI 2200 (FOCS)

I.

a)

p	q	r	q ∨ r	p ∧ (q ∨ r)	p ∧ q	p ∧ r	(p ∧ q) ∨ (p ∧ r)
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

b)

p	q	r	p ∧ q	(p ∧ q) ∨ r	¬ ((p ∧ q) ∨ r)
T	T	T	T	T	F
T	T	F	T	T	F
T	F	T	F	T	F
T	F	F	F	F	T
F	T	T	F	T	F
F	T	F	F	F	T
F	F	T	F	T	F
F	F	F	F	F	T

¬ p	¬ q	¬ r	¬ p ∧ ¬ r	¬ q ∧ ¬ r	(¬ p ∧ ¬ r) ∨ (¬ q ∧ ¬ r)
F	F	F	F	F	F
F	F	T	F	F	F
F	T	F	F	F	F
F	T	T	F	T	T
T	F	F	F	F	F
T	F	T	T	F	T
T	T	F	F	F	F
T	T	T	T	T	T

II. Predicate Logic Problems

You are given the following predicates defined on the domain of "everyone on the RPI campus":
$S(x)$ = "x is a student"; $W(x)$ = "x is wise"; $F(x, y)$ = "x is a friend of y". Use these predicates to formalize the following English sentences in predicate logic.

(a) Puckman is a student. # *Note: Just like Python, you may use a literal as function input.*

$S(Puckman)$

(b) No students are wise.

$\forall x(S(x) \rightarrow \neg W(x))$

or equivalently:

$\neg\exists x(S(x) \wedge W(x))$

(c) All wise students are friends with Puckman.

$\forall x((W(x) \wedge S(x)) \rightarrow F(x, \text{Puckman}))$

(d) There is exactly one student who is a friend of Dan's. # *Note: $x \neq y$ is a proposition you may find useful here.*

$\exists x(S(x) \wedge F(x, \text{Dan}) \wedge \forall y((y \neq x \rightarrow \neg(S(y) \wedge F(y, \text{Dan})))))$

Same thing as

$\exists x(S(x) \wedge F(x, \text{Dan}) \wedge \forall y((S(y) \wedge F(y, \text{Dan})) \rightarrow x = y))$

III

a) $m \land f \rightarrow A$
f == T / uncertain

b) $m \lor f \rightarrow A$
f == T / yes

c) $m \land f \rightarrow A$
A == T / yes

d) $m \lor f \rightarrow A$
A == T / uncertain

e) $m \land f \rightarrow A$
A == F / uncertain

f) $m \lor f \rightarrow A$
A == F / no

IV. Give direct proofs of the following statements:

(a) $(x \in \mathbb{Q} \wedge y \in \mathbb{Q}) \implies xy \in \mathbb{Q}$.

Proof. Let x and y be rational numbers. Then, by definition of rational numbers, there exist integers $a, b, c, d \in \mathbb{Z}$ with $b \neq 0$ and $d \neq 0$ such that

$$x = \frac{a}{b}, \quad y = \frac{c}{d}.$$

Their product is

$$xy = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Since ac and bd are integers and $bd \neq 0$, it follows that $\frac{ac}{bd}$ is a rational number. Hence xy is rational.

Therefore, whenever $x, y \in \mathbb{Q}$, their product xy is also in $\mathbb{Q}$. □

(b) $n \in \mathbb{N} \implies n^2 + n$ is even.

Proof. Let n be a natural number. We consider two cases:

Case 1: n is even. Then $n = 2k$ for some integer $k \in \mathbb{Z}$. Hence

$$n^2 + n = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k),$$

which is an even integer.

Case 2: n is odd. Then $n = 2k + 1$ for some integer $k \in \mathbb{N}_0$. Hence

$$n^2+n = (2k+1)^2+(2k+1) = 4k^2+4k+1+2k+1 = 4k^2+6k+2 = 2(2k^2+3k+1),$$

which is an even integer.

In both cases, $n^2 + n$ is even. Therefore, for every natural number n, the sum $n^2 + n$ is even. □

V. Prove by Contraposition

Prove the following by contraposition:

(a) If x is irrational, then $\sqrt{x}$ is irrational.

Proof by contraposition: We will prove that if $\sqrt{x}$ is rational, then x is rational.

- Assume $\sqrt{x}$ is rational.
- Then $\sqrt{x} = \frac{a}{b}$ for some integers a and b where $b \neq 0$.
- Therefore $x = (\sqrt{x})^2 = (\frac{a}{b})^2 = \frac{a^2}{b^2}$.
- Squares of integers are integers.
- Since a^2 and b^2 are integers and $b^2 \neq 0$, this shows that x is rational.
- Thus we have proven that if $\sqrt{x}$ is rational, then x is rational.
- By contraposition, this proves that if x is irrational, then $\sqrt{x}$ is irrational. ■

(b) $\forall m, n, d \in \mathbb{N}$, if mn is not divisible by d, then neither m nor n is divisible by d.

Proof by contraposition: We will prove that if either m or n is divisible by d, then mn is divisible by d.

- Assume either m or n is divisible by d.
- Case 1: If m is divisible by d, then $m = kd$ for some $k \in \mathbb{N}$.
  - Then $mn = (kd)n = d(kn)$
  - Since k and n are natural numbers, their product kn is also a natural number
  - Therefore $mn = d(kn)$ where $kn \in \mathbb{N}$, showing mn is divisible by d
- Case 2: If n is divisible by d, then $n = kd$ for some $k \in \mathbb{N}$.
  - Then $mn = m(kd) = d(mk)$
  - Since m and k are natural numbers, their product mk is also a natural number
  - Therefore $mn = d(mk)$ where $mk \in \mathbb{N}$, showing mn is divisible by d
- In both cases, we have shown that mn is divisible by d.
- Thus we have proven that if either m or n is divisible by d, then mn is divisible by d.
- By contraposition, this proves that if mn is not divisible by d, then neither m nor n is divisible by d. ■

(VI) Prove the following by contradiction:

(a) There is no smallest positive rational number.

Suppose there exists a smallest positive rational number $q^* = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b > 0$. Notice that since $q^* > 0$, we have $q^* \cdot \frac{1}{2} < q^*$.

Substituting $q^* = \frac{a}{b}$, we find that

$$\frac{q^*}{2} = \frac{a}{2b}$$

which is rational since $a, 2b \in \mathbb{Z}$ and $2b \neq 0$.

Thus, $\frac{q^*}{2}$ is a positive rational number smaller than q^*, contradicting the assumption that q^* is the smallest positive rational number.

Therefore we have shown that there is no smallest rational number by contradiction. ■

(b) $\log_2 9$ is irrational.

Suppose that $\log_2 9$ rational. Then, there exist $a, b \in \mathbb{Z}$ with $b > 0$ such that

$$\log_2 9 = \frac{a}{b}$$

Applying the definition of logarithms, we have

$$2^{a/b} = 9$$

Raising both sides to the power of b, we obtain

$$2^a = 9^b$$

However, 2^a is an even number, while 9^b is an odd number. Since an even number cannot equal an odd number, this leads to a contradiction.

Therefore we have shown that $\log_2 9$ is irrational by contradiction. ■