Recitation 11 solutions - CSCI 2200 (FOCS)

I. Let X be the waiting time with success probability p. Compute $f(n) = \mathbb{P}[X \geq n]$ and $g(n)$, the Chebyshev upper bound on $\mathbb{P}[X \geq n]$. Show that $f(n) = o(g(n))$. (Chebyshev's bound can be asymptotically worse than reality). For a waiting time with success probability p, we know:

$$\mathbb{P}[X = k] = p(1-p)^{k-1} \quad \text{for } k \geq 1$$

To compute $f(n) = \mathbb{P}[X \geq n]$:

$$\begin{aligned}
f(n) = \mathbb{P}[X \geq n] &= \sum_{k=n}^{\infty} \mathbb{P}[X = k] \\
&= \sum_{k=n}^{\infty} p(1-p)^{k-1} \\
&= p \sum_{k=n}^{\infty} (1-p)^{k-1} \\
&= p(1-p)^{n-1} \sum_{j=0}^{\infty} (1-p)^j \quad \text{(substituting } j = k-n) \\
&= p(1-p)^{n-1} \cdot \frac{1}{1-(1-p)} \quad \text{(using the formula for infinite geometric series)} \\
&= p(1-p)^{n-1} \cdot \frac{1}{p} \\
&= (1-p)^{n-1}
\end{aligned}$$

For Chebyshev's inequality, we need the mean and variance of X.

Mean: $\mathbb{E}[X] = \frac{1}{p}$

Variance: $\mathrm{Var}(X) = \frac{1-p}{p^2}$

Using Chebyshev's inequality for the deviation from the mean:

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq \frac{\mathrm{Var}(X)}{t^2}$$

For our case, we want $\mathbb{P}[X \geq n]$, so we set:

$$n - \frac{1}{p} = t$$

This gives us:

$$\begin{aligned} g(n) &= \mathbb{P}\left[X - \frac{1}{p} \geq n - \frac{1}{p}\right] \\ &\leq \mathbb{P}\left[\left|X - \frac{1}{p}\right| \geq n - \frac{1}{p}\right] \\ &\leq \frac{\mathrm{Var}(X)}{\left(n - \frac{1}{p}\right)^2} \\ &= \frac{\frac{1-p}{p^2}}{\left(n - \frac{1}{p}\right)^2} \\ &= \frac{1-p}{p^2\left(n - \frac{1}{p}\right)^2} \end{aligned}$$

This is only valid when $n > \frac{1}{p}$. For $n \leq \frac{1}{p}$, Chebyshev's inequality gives the bound $g(n) \leq 1$.

To show $f(n) = o(g(n))$, we need to show that:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

For large n:

$$\begin{aligned} \frac{f(n)}{g(n)} &= \frac{(1-p)^{n-1}}{\frac{1-p}{p^2\left(n-\frac{1}{p}\right)^2}} \\ &= \frac{(1-p)^{n-1} \cdot p^2\left(n - \frac{1}{p}\right)^2}{1-p} \\ &= p^2(1-p)^{n-2}\left(n - \frac{1}{p}\right)^2 \end{aligned}$$

As $n \to \infty$, the term $(1-p)^{n-2}$ approaches 0 exponentially, while $\left(n - \frac{1}{p}\right)^2$ grows only polynomially. Therefore:

$$\lim_{n \to \infty} p^2(1-p)^{n-2}\left(n - \frac{1}{p}\right)^2 = 0$$

This proves that $f(n) = o(g(n))$, showing that Chebyshev's bound can be asymptotically worse than the true probability.■

II. A couple has kids until five boys. Estimate $\mathbb{P}[7 \leq \text{number of children} \leq 13]$ using Chebyshev's inequality and compare with the true value.

Let X be the random variable representing the number of children until the fifth boy is born.

This follows a negative binomial distribution with parameters $r = 5$ (number of successes) and $p = \frac{1}{2}$ (probability of success, assuming equal probability of having a boy or girl).

The mean and variance of a negative binomial distribution are:

$$\begin{aligned}
\mathbb{E}[X] &= \frac{r}{p} = \frac{5}{1/2} = 10 \\
\mathrm{Var}(X) &= \frac{r(1-p)}{p^2} = \frac{5 \cdot 1/2}{(1/2)^2} = \frac{5 \cdot 1/2}{1/4} = 10
\end{aligned}$$

Using Chebyshev's inequality:

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq \frac{\mathrm{Var}(X)}{t^2}$$

For $\mathbb{P}[7 \leq X \leq 13]$, we can rewrite this as:

$$\begin{aligned}
\mathbb{P}[7 \leq X \leq 13] &= \mathbb{P}[|X - 10| \leq 3] \\
&= 1 - \mathbb{P}[|X - 10| > 3]
\end{aligned}$$

By Chebyshev's inequality:

$$\mathbb{P}[|X - 10| > 3] \leq \frac{\mathrm{Var}(X)}{3^2} = \frac{10}{9} \approx 1.11$$

Since a probability cannot exceed 1, we take $\mathbb{P}[|X - 10| > 3] \leq 1$.

Therefore:

$$\mathbb{P}[7 \leq X \leq 13] \geq 1 - 1 = 0$$

This doesn't give us a useful lower bound. Let's try a different approach using Chebyshev's one-sided inequality:

$$\mathbb{P}[X - \mu \geq t] \leq \frac{\sigma^2}{\sigma^2 + t^2}$$

For the lower tail:

$$\mathbb{P}[X < 7] = \mathbb{P}[X - 10 < -3] \leq \frac{10}{10 + 9} = \frac{10}{19} \approx 0.526$$

For the upper tail:

$$\mathbb{P}[X > 13] = \mathbb{P}[X - 10 > 3] \leq \frac{10}{10+9} = \frac{10}{19} \approx 0.526$$

Therefore:

$$\begin{aligned}
\mathbb{P}[7 \leq X \leq 13] &= 1 - \mathbb{P}[X < 7] - \mathbb{P}[X > 13] \\
&\geq 1 - \frac{10}{19} - \frac{10}{19} \\
&= 1 - \frac{20}{19} \\
&= -\frac{1}{19} \approx -0.053
\end{aligned}$$

This gives us a negative lower bound, which isn't helpful. This is because Chebyshev's inequality provides a loose bound.

For the true probability, we need to calculate $\mathbb{P}[7 \leq X \leq 13]$ using the negative binomial distribution:

$$\mathbb{P}[X = k] = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

For our case, $r = 5$ and $p = \frac{1}{2}$:

$$\mathbb{P}[X = k] = \binom{k-1}{4} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{k-5} = \binom{k-1}{4} \left(\frac{1}{2}\right)^k$$

Calculating for each value from $k = 7$ to $k = 13$ and summing:

$$\begin{aligned}
\mathbb{P}[7 \leq X \leq 13] = \sum_{k=7}^{13} \binom{k-1}{4} \left(\frac{1}{2}\right)^k \\
&= \binom{6}{4} \left(\frac{1}{2}\right)^7 + \binom{7}{4} \left(\frac{1}{2}\right)^8 + \ldots + \binom{12}{4} \left(\frac{1}{2}\right)^{13} \\
&= 15 \cdot 2^{-7} + 35 \cdot 2^{-8} + 70 \cdot 2^{-9} + 126 \cdot 2^{-10} + 210 \cdot 2^{-11} + 330 \cdot 2^{-12} + 495 \cdot 2^{-13} \\
&\approx 0.117 + 0.137 + 0.137 + 0.123 + 0.103 + 0.081 + 0.060 \\
&\approx 0.758
\end{aligned}$$

Therefore, the true probability is approximately 0.758 or about 75.8%, while Chebyshev's inequality didn't provide a useful bound for this problem.

III. 100 people toss their hats up. The hats land randomly on heads. Let the random variable $\mathbf{X}$ be the number of people who get their hats back.

(a) Compute $\mathbb{E}[\mathbf{X}]$ and $\text{var}[\mathbf{X}]$.

Wlog, assume we have n people. Denote $\mathbf{X}_i$ is the binary indicator of which express that i-th person getting their hat back.

Then we know that

$$\mathbb{P}[\mathbf{X}_i = 1] = \frac{1}{n}$$

Since we want $\mathbb{E}[\mathbf{X}]$, it is equal to $\mathbb{E}[\mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_n]$, So:

$$\mathbb{E}[\mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_n] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i]$$

To find $\mathbb{E}[\mathbf{X}_i]$,

$$\mathbb{E}[X_i] = 0 \times \mathbb{P}[X_i = 0] + 1 \times \mathbb{P}[X_i = 1]$$
$$\mathbb{E}[X_i] = \mathbb{P}[X_i = 1]$$
$$\mathbb{E}[X_i] = \frac{1}{n}$$

Thus,

$$\mathbb{E}[\mathbf{X}] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \frac{1}{n} = 1$$

Next, for $\text{var}[\mathbf{X}]$,

Since

$$\text{var}[\mathbf{X}] = \mathbb{E}[\mathbf{X}^2] - (\mathbb{E}[\mathbf{X}])^2,$$

and since

$$\mathbf{X}^2 = \sum_{i} \mathbf{X}_i^2 + \sum_{i,j:i \neq j} \mathbf{X}_i \mathbf{X}_j$$

We already know that $\mathbf{X}_i^2 = \frac{1}{n}$, so only for later part,

$$\mathbb{E}[\mathbf{X}_i\mathbf{X}_j] = \mathbb{P}[\mathbf{X}_i\mathbf{X}_j = 1] = \mathbb{P}[\mathbf{X}_i = 1,\ \mathbf{X}_j = 1] = \mathbb{P}[\mathbf{X}_i = 1]\mathbb{P}[\mathbf{X}_j = 1|\mathbf{X}_i = 1] = \frac{1}{n} \cdot \frac{1}{n-1}$$

Now, calculate the sum,

$$\mathbb{E}[\mathbf{X}^2] = n \cdot \frac{1}{n} + (n^2 - n) \cdot \frac{1}{n} \cdot \frac{1}{n-1} = 1 + 1 = 2$$

Since $(\mathbb{E}[\mathbf{X}])^2 = 1$, we have

$$\text{var}[\mathbf{X}] = \mathbb{E}[\mathbf{X}^2] - (\mathbb{E}[\mathbf{X}])^2 = 2 - 1 = 1$$

In summary, we have:

$$\mathbb{E}[\mathbf{X}] = 1$$

$$\text{var}[\mathbf{X}] = 1$$

(b) Give a (non-trivial) upper bound on the probability that more than half the people get their hats back.

To find an upper bound for $\mathbb{P}[X > 50]$, use Markov Inequality:

$$\mathbb{P}[\mathbf{X} \geq 50] \leq \frac{\mathbb{E}[\mathbf{X}]}{50}$$

which is:

$$\mathbb{P}[\mathbf{X} \geq 50] \leq 0.02$$

Or, use Chernoff bound.
If $\mathbb{E}[X] = \mu$, then for any $\delta > 0$:

$$\mathbb{P}\left(X \geq (1+\delta)\mu\right) \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}$$

We already know that $\mathbb{E}[\mathbf{X}] = \mu = 1$ and $(1+\delta)\mu = 50$ as a definition. Thus,

$$1+\delta = 50, \ \ \delta = 49$$

Then now we have:

$$\mathbb{P}[X \geq 50] \leq \left(\frac{e^{49}}{50^{50}}\right)^{1} = \frac{e^{49}}{50^{50}}$$

IV. An aggressive drunk takes 10 steps $\mathbf{X}_1, \ldots, \mathbf{X}_{10}$. Each step is independent, and moves left or right with equal probability, The size of the step increases with time, $[\mathbf{X}]_i = i$. Use linearity to compute the expected value and standard deviation of the drunk's position after 10 steps.

Assume the final position of drunk is $\mathbf{F}$. We need $\mathbb{E}[\mathbf{F}]$ first. So:

$$\mathbb{E}[\mathbf{F}] = \mathbb{E}\left[\sum_{i=0}^{10} \mathbf{X}_i\right] = \sum_{i=0}^{10} \mathbb{E}[\mathbf{X}_i]$$

For each step the drunk moves left or right with equal $p = \frac{1}{2}$, and move i steps.

$$\mathbb{E}[\mathbf{X}_i] = \frac{1}{2} \cdot i + \frac{1}{2} \cdot (-i) = 0$$

Thus, expected position of the drunk is 0.

For variance, we know that:

$$\text{var}[\mathbf{X}_i] = \mathbb{E}[\mathbf{X}_i^2] - (\mathbb{E}[\mathbf{X}_i])^2$$

Since $\mathbf{X}$ is independent we just merely calculate:

$$= \left[\frac{1}{2} \cdot i^2 + \frac{1}{2} \cdot (-i)^2\right] - 0 = i^2$$
$$\text{var}[\mathbf{X}_i] = i^2$$

For $\text{var}[\mathbf{F}]$,

$$\text{var}[\mathbf{F}] = \sum_{i=0}^{10} i^2 = \frac{10(10+1)(2 \cdot 10 + 1)}{6} = \frac{2310}{6} = 385$$

Since $\text{stdv} = \sqrt{\text{var}}$,

$$\sigma = \sqrt{385}$$

V. Prove that the unweighted finite graphs are countable. What about weighted graphs with integer weights? What if the weights are real?

Proof Let $\mathcal{G}_n$ be the collection of all graphs with n vertices. If we label the vertices $1, 2, \ldots, n$ then each graph has $\binom{n}{2}$ possible edges. For each edge, we have a binary choice as to whether that edge is in our graph which gives us a maximum of $2^{\binom{n}{2}}$ possible graphs of size n (this count is an upper bound as we overcount graphs that are isomorphic).

Now let $\mathcal{G}$ be the collection of all finite graphs. Notice that we can express $\mathcal{G}$ as a countable union of collections as follows

$$\mathcal{G} = \bigcup_{i=0}^{\infty} \mathcal{G}_i$$

This union is countable because there is a clear bijection from $\{\mathcal{G}_n \mid n \in \mathbb{N}_0\}$ and $\mathbb{N}_0$ which takes the collection $\mathcal{G}_n$ to n. Since we have already shown that each collection $\mathcal{G}_n$ is finite, it follows that $\mathcal{G}$ is a countable union of finite sets which makes $\mathcal{G}$ a countable set.

Therefore we have shown the set of unweighted finite graphs are countable. ■

Let's now consider graphs with integer edge weights. Notice that each finite graph $G = (V, E)$ can be given a vector of weights $\vec{w} \in \mathbb{Z}^{|E|}$. Since the set $\mathbb{Z}^{|E|}$ is a finite Cartesian product of countable sets, the set is countable which means that for each graph, there are a countable number of ways to ascribe integer edge weights.

Since we have shown that there are a countable number of unweighted graphs and for each unweighted graph, there are a countable number of weights that we can ascribe, it follows that set of all integer-weighted graphs is a countable collection of countable sets which is countable.

This argument relies on the fact that the integers are countable and hence a finite Cartesian product of integers is countable. When we allow the edges to be arbitrary real values, the set of all weights for a given graph $G = (V, E)$ becomes $\mathbb{R}^{|E|}$ which is uncountable. Therefore the number of real-weighted graphs is a countable union of uncountable sets which must be uncountable.

VI. The positive rationals are $\mathbb{Q}_+ = \left\{\frac{x}{y} \mid x, y \in \mathbb{N}\right\}$, We will use $\mathbb{Q}_+$ to prove that $\mathbb{Q}$ is countable.

(a) Argue that $|\mathbb{Q}_+| \leq |\mathbb{N}|$ by establishing that $f(x, y) = 2^x 3^y$ is an injection from $\mathbb{Q}_+$ to $\mathbb{N}$

Proof We will show that $|\mathbb{Q}_+| \leq |\mathbb{N}|$ by showing that the map $f : \mathbb{Q}_+ \to \mathbb{N}$ that maps a rational number $\dfrac{x}{y}$ in lowest terms to $2^x 3^y$ is an injection.

Let $\dfrac{x}{y}$ and $\dfrac{x'}{y'}$ be rationals in reduced form such that $2^x 3^y = 2^{x'} 3^{y'}$. To show that our map is injective, we must show that $\dfrac{x}{y} = \dfrac{x'}{y'}$. By properties of exponents, it follows that

$$2^{x-x'} = 3^{y'-y}$$

Since 2 and 3 are coprime, the only way in which an integer powers of 2 and 3 can be equal are if both exponents are zero. This means that $x - x' = 0$ and $y' - y = 0$ which tells us that $x = x'$ and $y = y'$ which means that $\dfrac{x}{y} = \dfrac{x'}{y'}$

Since $\dfrac{x}{y}$ and $\dfrac{x'}{y'}$ are arbitrary rational numbers, it we have shown that our map f is an injection from $\mathbb{Q}_+$ to $\mathbb{N}$ and thus $|\mathbb{Q}_+| \leq |\mathbb{N}|$. ■

(b) Show that $\mathbb{Q}_+$ being countable means that $\mathbb{Q}$ is countable.

Proof Assuming that $\mathbf{Q}_+$ is countable, we will show that $\mathbb{Q}$ is countable by showing that there exists some injection $g : \mathbb{Q} \to \mathbb{Q}_+$.

Let $g : \mathbb{Q} \to \mathbb{Q}_+$ such that

$$g(x) = \begin{cases} x+1 & \text{if } x \geq 0 \\ \frac{1}{1-x} & \text{otherwise} \end{cases}$$

We wish to show that g is an injection.

Let $x, y \in \mathbb{Q}$ such that $g(x) = g(y)$. To show that $x = y$, we will consider three cases:

$$(1)\ \textit{(WLOG)}\ x \geq 0 \land y < 0, \qquad (2)\ x \geq 0 \land y \geq 0, \qquad (3)\ x < 0 \land y < 0$$

[**Case 1**] Without loss of generality, suppose that $x \geq 0$ and $y < 0$. Since $x \geq 0$, it follows that $g(x) = x+1 \geq 1$. Likewise since $y < 0$, it follows that $g(y) = \dfrac{1}{1-y} < 1$. However, since $g(x) = g(y)$, it must follow that $g(x) < 1$ and $g(x) \geq 1$, but this is clearly a contradiction. Therefore this case cannot happen and either both values are greater than or equal to zero or less than zero.

[**Case 2**] Now suppose that $x \geq 0$ and $y \geq 0$. This means that

$$\begin{aligned} g(x) &= g(y) \\ x+1 &= y+1 \\ x &= y \end{aligned}$$

[**Case 3**] Finally, suppose that $x < 0$ and $y < 0$. Therefore,

$$\begin{aligned} g(x) &= g(y) \\ \frac{1}{1-x} &= \frac{1}{1-y} \\ 1-x &= 1-y \\ x &= y \end{aligned}$$

Since we have shown that in every valid case that for any arbitrary $x, y \in \mathbb{Q}$, if $g(x) = g(y)$ then $x = y$, we have shown that our map g is an injection from $\mathbb{Q}$ to $\mathbb{Q}_+$.

Thus we have shown that $|\mathbb{Q}| \leq |\mathbb{Q}_+|$ which means that $\mathbb{Q}$ is countable since we have already shown that $|\mathbb{Q}_+|$ is countable. ■

VII. Solution

Claim: The set of eventually constant infinite sequences on $\mathbb{N}$ is countable.

Proof:

To show that a set A is countable, we must find a 1-to-1 mapping from A into $\mathbb{N}$ or from $\mathbb{N}$ onto A.

Consider the set of eventually constant sequences on $\mathbb{N}$. Each sequence in this set is completely determined by a finite initial segment followed by one repeated constant value.

The set of all finite sequences of natural numbers is countable since we can list finite sequences first by length (which is finite), and within each length, enumerate all possible combinations (finite products of countable sets are countable).

Each eventually constant sequence can be represented by pairing:

1. A finite sequence of natural numbers (countable),
2. A single natural number indicating the repeated constant (countable).

Since a countable union of countable sets is countable, the set of all eventually constant infinite sequences is also countable.

$$\boxed{\text{Eventually constant infinite sequences on } \mathbb{N} \text{ are countable.}}$$

VIII. Solution

Claim: The set of increasing sequences on $\mathbb{N}$ has the same cardinality as $\mathbb{R}$.

Proof:

An increasing sequence on $\mathbb{N}$ corresponds exactly to an infinite subset of $\mathbb{N}$. Thus, the cardinality of the set of all increasing sequences is equal to the cardinality of the power set of $\mathbb{N}$, which is the set of all subsets of $\mathbb{N}$.

According to Cantor's diagonal argument (as shown in the slides), the set of all infinite binary strings (and equivalently, all subsets of $\mathbb{N}$) is uncountable. Furthermore, it is known from the slides that the cardinality of the set of infinite binary strings (subsets of $\mathbb{N}$) is exactly the cardinality of $\mathbb{R}$.

Therefore, the set of increasing sequences on $\mathbb{N}$ has the same cardinality as the real numbers, $\mathbb{R}$.

$$\boxed{\text{Increasing infinite sequences on } \mathbb{N} \text{ have cardinality equal to } |\mathbb{R}|.}$$