Recitation 10 Solutions

I. A random variable $\mathbf{X}$ takes values in $\{0, 1, 2 \ldots\}$. Show that

$$\mathbb{E}[\mathbf{X}] = \sum_{x=1}^{\infty} \mathbb{P}[\mathbf{X} \geq x] = \sum_{x=0}^{\infty} \mathbb{P}[\mathbf{X} > x]$$

This result is called the *tail sum formula* and is very useful.

Proof Recall the standard formula for expected value as given below:

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} n\mathbb{P}[\mathbf{X} = n]$$

Since $n \in \mathbb{N}$ and each term in our sum has n copies of $\mathbb{P}[\mathbf{X} = n]$, we can rewrite our sum as follows:

$$\begin{aligned}\sum_{n=1}^{\infty} n\mathbb{P}[\mathbf{X} = n] &= \sum_{n=1}^{\infty}\sum_{x=1}^{n} \mathbb{P}[\mathbf{X} = n] \\ &= \mathbb{P}[\mathbf{X} = 1] + \\ &\quad \mathbb{P}[\mathbf{X} = 2] + \mathbb{P}[\mathbf{X} = 2] + \\ &\quad \mathbb{P}[\mathbf{X} = 3] + \mathbb{P}[\mathbf{X} = 3] + \mathbb{P}[\mathbf{X} = 3] + \\ &\quad \ldots\end{aligned}$$

Notice that in this expression we are summing up each row first and then summing up the columns. Notice that we get an equivalent expression if we instead sum up the columns first and then sum up the rows. This changes our sum to

$$\sum_{n=1}^{\infty}\sum_{x=1}^{n} \mathbb{P}[\mathbf{X} = n] = \sum_{x=1}^{\infty}\sum_{n=x}^{\infty} \mathbb{P}[\mathbf{X} = n]$$

If you are taking or have taken multivariable calculus, you may recognize this as a discrete parallel to changing the order of integration of a double integral.

Since the inner sum is adding up all the probabilities for all cases when $\mathbf{X} = n \geq x$, we know that

$$\sum_{x=1}^{\infty}\sum_{n=x}^{\infty} \mathbb{P}[\mathbf{X} = n] = \sum_{x=1}^{\infty} \mathbb{P}[\mathbf{X} \geq x]$$

which gives us our first result.

To get our second result, we can make some algebraic manipulations to our first result as follows:

$$\begin{aligned}\sum_{x=1}^{\infty} \mathbb{P}[\mathbf{X} \geq x] &= \sum_{x=1}^{\infty} (\mathbb{P}[\mathbf{X} > x] + \mathbb{P}[\mathbf{X} = x]) \\ &= \sum_{x=1}^{\infty} \mathbb{P}[\mathbf{X} > x] + \sum_{x=1}^{\infty} \mathbb{P}[\mathbf{X} = x] \\ &= \sum_{x=1}^{\infty} \mathbb{P}[\mathbf{X} > x] + \mathbb{P}[\mathbf{X} > 0] \\ &= \sum_{x=0}^{\infty} \mathbb{P}[\mathbf{X} > x]\end{aligned}$$

Thus we have shown that both forms of the tail sum formula are equivalent ways of calculating expected value. ■

Recitation 10 Solutions

II. A team is equally likely to win or lose its first game. In each following game, the previous result is twice as likely as the opposite result. What is the expected number of games played to get two wins?

Let $\mathbf{X}_i$ = number of games played to get two wins and let $E = \mathbb{E}[\mathbf{X}_2]$. Notice that after we play one game, since the probability of winning and losing are both $\frac{1}{2}$,

$$E = 1 + \frac{1}{2}\mathbb{E}[\mathbf{X}_1 \mid \text{last game won}] + \frac{1}{2}\mathbb{E}[\mathbf{X}_2 \mid \text{last game lost}]$$

We will now find $\mathbb{E}[\mathbf{X}_1 \mid \text{last game won}]$ by doing a similar expansion:

$$\mathbb{E}[\mathbf{X}_1 \mid \text{last game won}] = 1 + \frac{2}{3}\mathbb{E}[\mathbf{X}_0 \mid \text{last game won}] + \frac{1}{3}\mathbb{E}[\mathbf{X}_1 \mid \text{last game lost}]$$

where $\mathbb{E}[\mathbf{X}_0 \mid \text{last game won}] = 0$ (since we have already reached the goal) and $\mathbb{E}[\mathbf{X}_1 \mid \text{last game lost}]$ expands as follows:

$$\mathbb{E}[\mathbf{X}_1 \mid \text{last game lost}] = 1 + \frac{1}{3}\mathbb{E}[\mathbf{X}_0 \mid \text{last game won}] + \frac{2}{3}\mathbb{E}[\mathbf{X}_1 \mid \text{last game lost}]$$

If we plug in 0 for $\mathbb{E}[\mathbf{X}_0 \mid \text{last game won}]$, we can solve for $\mathbb{E}[\mathbf{X}_1 \mid \text{last game lost}]$

$$\begin{aligned}
\mathbb{E}[\mathbf{X}_1 \mid \text{last game lost}] &= 1 + \frac{2}{3}\mathbb{E}[\mathbf{X}_1 \mid \text{last game lost}] \\
3\mathbb{E}[\mathbf{X}_1 \mid \text{last game lost}] &= 3 + 2\mathbb{E}[\mathbf{X}_1 \mid \text{last game lost}] \\
\mathbb{E}[\mathbf{X}_1 \mid \text{last game lost}] &= 3
\end{aligned}$$

This tells us that

$$\mathbb{E}[\mathbf{X}_1 \mid \text{last game won}] = 1 + \frac{3}{3} = 2$$

Now we can solve for $\mathbb{E}[\mathbf{X}_2 \mid \text{last game lost}]$ using similar expansions.

$$\begin{aligned}
\mathbb{E}[\mathbf{X}_2 \mid \text{last game lost}] &= 1 + \frac{1}{3}\mathbb{E}[\mathbf{X}_1 \mid \text{last game won}] + \frac{2}{3}\mathbb{E}[\mathbf{X}_2 \mid \text{last game lost}] \\
&= 1 + \frac{2}{3} + \frac{2}{3}\mathbb{E}[\mathbf{X}_2 \mid \text{last game lost}]
\end{aligned}$$

Solving this equation tells us that $\mathbb{E}[\mathbf{X}_2 \mid \text{last game lost}] = 5$.

Putting everything together, we have that

$$\begin{aligned}
E &= 1 + \frac{1}{2}\mathbb{E}[\mathbf{X}_1 \mid \text{last game won}] + \frac{1}{2}\mathbb{E}[\mathbf{X}_2 \mid \text{last game lost}] \\
&= 1 + \frac{5}{2} + \frac{2}{2} \\
&= \frac{9}{2}
\end{aligned}$$

Therefore, the expected number of games needed to get two wins is $\boxed{4.5}$.

III. A gambler walks into a casino with \$50 and plays roulette. The gambler bets \$1 on red (probability $\frac{18}{36}$ to win) and keeps betting until either going bankrupt or doubling his money. If it takes about 1 minute to play one game of roulette, how many hours of entertainment does the gambler expect to have?

Let $E_i = \mathbb{E}\left[\text{minutes of entertainment left} \mid \text{gambler has } i \text{ dollars}\right]$. Using the law of total expectation and the general recurrence for the gambler's ruin problem, we have that

$$E_i = 1 + \frac{1}{2}E_{i-1} + \frac{1}{2}E_{i+1}$$

We can then write this as a difference equation where we define $\Delta E_i = E_{i+1} - E_i$

$$\begin{aligned} 2E_{i+1} &= 2 + E_i + E_{i-2} \\ (E_{i+2} - E_{i+1}) - (E_{i+1} - E_i) &= -2 \\ \Delta E_{i+1} - \Delta E_i &= -2 \\ \Delta^2 E_i &= -2 \end{aligned}$$

Using the Fundamental Theorem of discrete calculus, we can find a closed form for E_i

$$\begin{aligned} \Delta^2 E_i &= -2 \\ \sum_{j=0}^{k-1}\sum_{i=0}^{j-1} \Delta^2 E_i &= -2\sum_{j=0}^{k-1}\sum_{i=0}^{j-1} 1 \\ \sum_{j=0}^{k-1}(\Delta E_j - \Delta E_0) &= -2\sum_{j=0}^{k-1} j \\ E_k - E_0 - k\Delta E_0 &= -k(k-1) \\ E_k &= E_0 + k(\Delta E_0 + 1 - k) \end{aligned}$$

Notice that by the problem statement, if $i = 0$ or $i = 100$, $E_i = 0$. We can now use this information to solve for E_i

$$\begin{aligned} E_i &= k(\Delta E_0 + 1 - k) \\ &= k(E_1 - E_0 + 1 - k) \\ &= k(E_1 + 1 - k) \end{aligned}$$

We can now use the fact that $E_{100} = 0$ to solve for E_1

$$\begin{aligned} 0 &= E_{100} \\ &= 100(E_1 + 1 - 100) \\ E_1 &= 99 \end{aligned}$$

Therefore, we have that

$$E_i = i(100 - i)$$

When we plug in $i = 50$, we get that

$$E_{50} = 50 \cdot 50 = 2500 \text{ minutes} \approx \boxed{41.6 \text{ hours}}$$

IV. Pick random numbers from $\{1, \ldots, 100\}$ with replacement until their sum exceeds 100. How many numbers do you expect to pick?

Let $\mathbf{X}_k$ be the number of numbers picked from $\{1, \ldots, 100\}$ with replacement until their sum exceeds k. Using the law of total expectation it follows that

$$\begin{aligned}
\mathbb{E}[\mathbf{X}_k] &= \sum_{i=1}^{100} \mathbb{E}[\mathbf{X}_k \mid i \text{ is picked first}]\, \mathbb{P}[i \text{ is picked first}] \\
&= \frac{1}{100} \sum_{i=1}^{100} (1 + \mathbb{E}[\mathbf{X}_{k-i}]) \\
&= 1 + \frac{1}{100} \sum_{i=1}^{k-1} \mathbb{E}[\mathbf{X}_{k-i}] + \frac{1}{100} \sum_{i=k}^{100} \mathbb{E}[\mathbf{X}_{k-i}] \\
&= 1 + \frac{1}{100} \sum_{i=0}^{k-1} \mathbb{E}[\mathbf{X}_i]
\end{aligned}$$

From this, we find a closed form solution for $\mathbb{E}[\mathbf{X}_k]$ finding the forward difference of $\mathbb{E}[\mathbf{X}_k]$.

$$\begin{aligned}
\mathbb{E}[\mathbf{X}_{k+1}] - \mathbb{E}[\mathbf{X}_k] &= 1 + \frac{1}{100} \sum_{i=0}^{k} \mathbb{E}[\mathbf{X}_i] - \left(1 + \frac{1}{100} \sum_{i=0}^{k-1} \mathbb{E}[\mathbf{X}_i]\right) \\
&= \frac{1}{100} \sum_{i=0}^{k-1} \mathbb{E}[\mathbf{X}_i] + \frac{1}{100}\mathbb{E}[\mathbf{X}_k] - \frac{1}{100} \sum_{i=0}^{k-1} \mathbb{E}[\mathbf{X}_i] \\
&= \frac{1}{100}\mathbb{E}[\mathbf{X}_k]
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\mathbf{X}_{k+1}] &= \frac{101}{100}\mathbb{E}[\mathbf{X}_k] \\
\mathbb{E}[\mathbf{X}_k] &= c\left(\frac{101}{100}\right)^k
\end{aligned}$$

Using the fact that $\mathbb{E}[\mathbf{X}_0] = 1$ (since we start off less than zero and surpass zero after picking any number), we know that

$$1 = \mathbb{E}[\mathbf{X}_0] = c\left(\frac{101}{100}\right)^0 = c$$

Therefore it follows that

$$\mathbb{E}[\mathbf{X}_k] = \left(\frac{101}{100}\right)^k$$

which means that

$$\mathbb{E}[\mathbf{X}_{100}] = \left(\frac{101}{100}\right)^{100} \approx \boxed{2.7048}$$

Recitation 10 Solutions

V. A bag has 4 balls of different colors. At each step, pick two random balls and paint one ball the other ball's color. Replace the balls and repeat. On average, how many steps will it take until all balls are the same color?

Let's call the balls in the bag a, b, c, and d. Without loss of generality, there are five states that our bag can be in as shown below:

(1) 4 distinct balls (a, b, c, d)
(2) 3 distinct balls (a, b, c, c)
(3) 2 distinct balls with 2 of each ball (a, a, b, b)
(4) 2 distinct balls with 3 of one ball and 1 of the other (a, a, a, b)
(5) 1 distinct ball (a, a, a, a)

Given our case numberings above, let $E_i = \mathbb{E}\,[\text{steps until all balls are the same color} \mid \text{bag is in case } i]$. Notice that $E_5 = 0$ since all balls are already the same color.

To calculate E_4, notice that there are three possible state changes:

- $4 \to 5$ if we choose (b, a) (probability $\frac{1}{4} \cdot \frac{3}{3} = \frac{1}{4}$)
- $4 \to 4$ if we choose (a, a) (probability $\frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$)
- $4 \to 3$ if we choose (a, b) (probability $\frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$)

Since we must pick a ball before transitioning states, we have that

$$E_4 = 1 + \frac{1}{4}E_5 + \frac{1}{2}E_4 + \frac{1}{4}E_3$$

Likewise, we can calculate E_3 by considering two possible state changes:

- $3 \to 4$ if we choose (b, a) or (a, b) (probability $2 \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{3}$)
- $3 \to 3$ if we choose (a, a) or (b, b) (probability $2 \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3}$)

$$E_3 = 1 + \frac{2}{3}E_4 + \frac{1}{3}E_3$$

We can now set up a system of equations and solve for E_4 and E_3 giving $E_4 = 5.5$ and $E_3 = 7$.

We have three state transitions for E_2:

- $2 \to 4$ if we choose (c, a) or (c, b) (probability $2 \cdot \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{3}$)
- $2 \to 3$ if we choose (a, b) or (b, a) (probability $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$)
- $2 \to 2$ if we choose $(c, c), (b, c)$, or (a, c) (probability $\frac{1}{2} \cdot \frac{3}{3} = \frac{1}{2}$)

$$E_2 = 1 + \frac{1}{3}E_4 + \frac{1}{6}E_3 + \frac{1}{2}E_2$$

Using the values for E_4 and E_3, we know deduce that $E_2 = 8$.

Finally, to calculate E_1, notice that no matter what two balls we pick, the bag will transition to state 2. Therefore

$$E_1 = 1 + E_2 = 9$$

This means that the average number of steps needed until all balls are the same color is $\boxed{9}$.

VI. A fair coin is tossed repeatedly. Find the expected number of flips until HHH appears.

Let C be the number of flips until HHH appears. We can use the law of total expectation to solve for $\mathbb{E}[C]$

$$\begin{aligned}\mathbb{E}[C] &= 1 + \mathbb{E}[C|H]\,\mathbb{P}[H] + \mathbb{E}[C|T]\,\mathbb{P}[T] \\ &= 1 + \frac{1}{2}\mathbb{E}[C|H] + \frac{1}{2}\mathbb{E}[C]\end{aligned}$$

This follows since the expected number of flips needed when the previous flip was tails is the same as the expected number of flips needed given no previous flips (since we still need three consecutive heads).

We can find similar formulations for $\mathbb{E}[C|H]$ and $\mathbb{E}[C|HH]$.

$$\begin{aligned}\mathbb{E}[C|H] &= 1 + \frac{1}{2}\mathbb{E}[C|HH] + \frac{1}{2}\mathbb{E}[C] \\ \mathbb{E}[C|HH] &= 1 + \frac{1}{2}\mathbb{E}[C|HHH] + \frac{1}{2}\mathbb{E}[C]\end{aligned}$$

Now notice that $\mathbb{E}[C|HHH] = 0$ since the sequence HHH has already appeared.

We can now solve for $\mathbb{E}[C]$:

$$\begin{aligned}\mathbb{E}[C] &= 1 + \frac{1}{2}\mathbb{E}[C|H] + \frac{1}{2}\mathbb{E}[C] \\ &= 1 + \frac{1}{2}\left(1 + \frac{1}{2}\mathbb{E}[C|HH] + \frac{1}{2}\mathbb{E}[C]\right) + \frac{1}{2}\mathbb{E}[C] \\ &= 1 + \frac{1}{2}\left(1 + \frac{1}{2}\left(1 + \frac{1}{2}\mathbb{E}[C|HHH] + \frac{1}{2}\mathbb{E}[C]\right) + \frac{1}{2}\mathbb{E}[C]\right) + \frac{1}{2}\mathbb{E}[C] \\ &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\mathbb{E}[C] + \frac{1}{4}\mathbb{E}[C] + \frac{1}{2}\mathbb{E}[C] \\ &= \frac{7}{4} + \frac{7}{8}\mathbb{E}[C] \\ \frac{1}{8}\mathbb{E}[C] &= \frac{7}{4} \\ \mathbb{E}[C] &= 14\end{aligned}$$

Therefore, the expected number of flips before HHH appears is $\boxed{14}$.

VII. Prove that every graph $G = (V, E)$ has a cut of size at least $|E|/2$. (A cut partitions V into two disjoint sets A, B; its size is the number of edges crossing from A to B).

(a) Construct the sets A, B by randomly placing vertices independently into one of the sets. Let $e = (u, v)$ be an edge in the graph. Compute $\mathbb{P}[u \text{ and } v \text{ are in different sets}]$.

To find $\mathbb{P}[u \text{ and } v \text{ are in different sets}]$, we can use the law of total probabilities:

$$\mathbb{P}[u \text{ and } v \text{ are in different sets}] = \sum_{k=0}^{|V|} \mathbb{P}[u \text{ and } v \text{ are in different sets} \mid |A| = k]\, \mathbb{P}[|A| = k]$$

If u and v are in two different sets, we have two disjoint cases: either $u \in A$ and $v \notin A$ or $u \notin A$ and $v \in A$. This gives us a formula to calculate $\mathbb{P}[u \text{ and } v \text{ are in different sets} \mid |A| = k]$:

$$\begin{aligned}\mathbb{P}[u \text{ and } v \text{ are in different sets} \mid |A| = k] &= \mathbb{P}[u \in A \wedge v \notin A \mid |A| = k] + \mathbb{P}[u \notin A \wedge v \in A \mid |A| = k] \\ &= \frac{k}{|V|} \cdot \frac{|V| - k}{|V| - 1} + \frac{|V| - k}{|V|} \cdot \frac{k}{|V| - 1} \\ &= \frac{2k(|V| - k)}{|V|(|V| - 1)}\end{aligned}$$

To find $\mathbb{P}[|A| = k]$, we can use the fact that for each vertex v, that vertex has probability $\frac{1}{2}$ of being in A. Therefore we can think of each vertex as a trial and think of $|A|$ the number of successes in $|V|$ trials with $p = \frac{1}{2}$. This tells us that

$$\mathbb{P}[|A| = k] = \frac{1}{2^{|V|}} \binom{|V|}{k}$$

This tells us that

$$\begin{aligned}\mathbb{P}[u \text{ and } v \text{ are in different sets}] &= \sum_{k=0}^{|V|} \binom{|V|}{k} \frac{2k(|V| - k)}{2^k |V|(|V| - 1)} \\ &= \frac{2^{1-|V|}}{|V|(|V| - 1)} \sum_{k=0}^{|V|} \binom{|V|}{k} k(|V| - k) \\ &= \frac{2^{1-|V|}}{|V|(|V| - 1)} \left(|V| \sum_{k=0}^{|V|} \binom{|V|}{k} k - \sum_{k=0}^{|V|} \binom{|V|}{k} k^2 \right)\end{aligned}$$

We can use the identity that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

$$\begin{aligned}\mathbb{P}[u \text{ and } v \text{ are in different sets}] &= \frac{2^{1-|V|}}{|V|(|V| - 1)} \left(|V| \sum_{k=0}^{|V|} \binom{|V|}{k} k - \sum_{k=0}^{|V|} \binom{|V|}{k} k^2 \right) \\ &= \frac{2^{1-|V|}}{|V|(|V| - 1)} \left(|V|^2 \sum_{k=0}^{|V|} \binom{|V| - 1}{k - 1} - |V| \sum_{k=0}^{|V|} \binom{|V| - 1}{k - 1} k \right) \\ &= \frac{2^{1-|V|}}{|V|(|V| - 1)} \left(2^{|V|-1} |V|^2 - |V| \sum_{k=0}^{|V|} \binom{|V| - 1}{k - 1} k \right) \\ &= 2^{1-|V|} \left(2^{|V|-1} - 2^{|V|-2} \right) \\ &= \frac{1}{2}\end{aligned}$$

This tells us that $\mathbb{P}[u \text{ and } v \text{ are in different sets}] = \frac{1}{2}$.

(b) Define the indicator $\mathbf{X}(e) = 1$ if u and v are in different sets. Show that the value of the cut is $\mathbf{X} = \sum_{e \in E} \mathbf{X}(e)$. Compute $\mathbb{E}[\mathbf{X}]$ and prove the claim.

To show that $\mathbf{X}$ is the value of the cut, notice that $\mathbf{X}$ is the sum of the indicators that are 1 which is the same as counting the number of distinct edges that connect vertices in two different sets. But an edge is part of our cut if and only if it connects vertices in two different sets which means that $\mathbf{X}$ must be the value of the cut.

Now we can solve for $\mathbb{E}[\mathbf{X}]$.

$$\begin{aligned}
\mathbb{E}[\mathbf{X}] &= \mathbb{E}\left[\sum_{e \in E} \mathbf{X}(e)\right] \\
&= \sum_{e \in E} \mathbb{E}[\mathbf{X}(e)] \\
&= \sum_{e \in E} \mathbb{P}[u \text{ and } v \text{ are in different sets}] \\
&= \sum_{e \in E} \frac{1}{2} \\
&= \frac{|E|}{2}
\end{aligned}$$

Therefore, since the expected value of an arbitrary cut is $\frac{|E|}{2}$, there must be some cut with size at least $\frac{|E|}{2}$. ■

VIII. Let $\mathbf{X}$ be the wait for k successes with success probability p. Compute $\mathbb{E}[\max(0, r - \mathbf{X})]$ for $r \geq k$.

(a) Let $\mathbf{X}$ have pdf $P_{\mathbf{X}}(i)$ for $i \geq k$. Show that $\mathbb{E}[\max(0, r - \mathbf{X})] = \sum_{i=k}^{r}(r-i)P_{\mathbf{X}}(i)$

Using the definition of expected value,

$$\begin{aligned}\mathbb{E}[\max(0, r - \mathbf{X})] &= \sum_{j=0}^{r} j\mathbb{P}[\max(0, r-\mathbf{X}) = j] \\ &= \sum_{j=1}^{r} j\mathbb{P}[\max(0, r-\mathbf{X}) = j]\end{aligned}$$

Since we are only adding up cases where $\max(0, r - \mathbf{X}) \geq 1$, we know that $\max(0, r - \mathbf{X}) = r - \mathbf{X}$

$$\begin{aligned}\mathbb{E}[\max(0, r - \mathbf{X})] &= \sum_{j=1}^{r} j\mathbb{P}[r - \mathbf{X} = j] \\ &= \sum_{j=1}^{r} j\mathbb{P}[\mathbf{X} = r - j]\end{aligned}$$

By letting $i = r - j$, we have that

$$\begin{aligned}\mathbb{E}[\max(0, r - \mathbf{X})] &= \sum_{i=k}^{r} (r-i)\mathbb{P}[\mathbf{X} = i] \\ &= \sum_{i=k}^{r} (r-i)P_{\mathbf{X}}(i).\end{aligned}$$

(b) Establish that $P_{\mathbf{X}}(i) = \binom{i-1}{k-1}p^k(1-p)^{i-k}$.

We proved this identity in problem VII of last recitation, but I will briefly go through the derivation again.

Notice that $P_{\mathbf{X}}(i)$ is calculating the probability that we wait i trials for k successes with probability p, This tells us trial i must be a success and the $i-1$ trials must have exactly $k-1$ successes.

Since there are $\binom{i-1}{k-1}$ ways to arrange $k-1$ successes in $i-1$ trials, we have that

$$P_{\mathbf{X}}(i) = \binom{i-1}{k-1}p^k(1-p)^{i-k}$$

(c) Hence, show that

$$\mathbb{E}[\max(0, r - \mathbf{X})] = \begin{cases} r & k = 0; \\ \sum_{i=k}^{r}(r-i)\binom{i-1}{k-1}p^k(1-p)^{i-k} & k > 0 \end{cases}$$

Notice that if $k = 0$ then our value of $\mathbf{X}$ is guaranteed to be 0. Therefore, when $k = 0$, $\mathbb{E}[\max(0, r - \mathbf{X})] = \mathbb{E}[\max(0, r)] = \max(0, r) = r$ since r is always non-negative.

Otherwise, when $k > 0$, we have already shown that the given formula holds. Therefore,

$$\mathbb{E}[\max(0, r - \mathbf{X})] = \begin{cases} r & k = 0; \\ \sum_{i=k}^{r}(r-i)\binom{i-1}{k-1}p^k(1-p)^{i-k} & k > 0 \end{cases}$$

■