Homework 9

Instructions: Standard course policies about typesetting, file size, and submission apply. You must show your work to receive credit. Your work must be your own, though you are permitted to get assistance from classmates or instructional staff. Your responses to the submission problems must be uploaded to Submitty by **8:59pm on Thursday, March 20**.

Attribution: these problems were chosen from the DMC text.

Note: you can employ Monte Carlo simulations to check some of your answers. But of course your arguments must be sound and not reference any simulations.

Recitation Problems

I. Suppose that A and B are independent.

(a) Which of the following pairs of events are independent?
(i) $A, \overline{B}$ (ii) $\overline{A}, \overline{B}$ (iii) $\overline{A}, B$ (iv) $A, \overline{A} \cap B$ (v) $A, \overline{A} \cup B$
(b) Show that: (i) $\mathbb{P}[A \mid \overline{B}] = \mathbb{P}[A]$, (ii) $\mathbb{P}[\overline{A} \mid B] = \mathbb{P}[\overline{A}]$.
(c) If A and B have strictly positive probabilities, can they be disjoint events?
(d) Can $\mathbb{P}[A] = 0$?
(e) Show that $\mathbb{P}[\overline{A \cup B}] = \mathbb{P}[\overline{A}] \times \mathbb{P}[\overline{B}]$.

II. A 90-sided die with faces $1, \ldots, 90$ is rolled 6 times. Compute the probability that all rolls are different.

III. Joan and Tariq try to access a database at time steps $1, 2, 3, \ldots$. If both try to access the database, both get locked out for that time step. Joan and Tariq implement a randomized algorithm. Each independently attempts to access the database with probability p (independently at every time step). Let

$$J(i) = \mathbb{P}[\text{Joan gains access to the database at time step } i].$$

Similarly define $T(i)$ for Tariq. Let $A(i)$ be the probability that one of them gains access at time step i.

(i) Compute $J(i)$, $T(i)$, and $A(i)$. Set p to the value that maximizes $J(i)$.
(ii) Show that $\mathbb{P}[\text{Joan waits } k \text{ steps for access}] = (\frac{3}{4})^{k-1}\frac{1}{4}$.
(iii) Show that $\mathbb{P}[\text{First successful access is after } k \text{ steps}] = (\frac{1}{2})^k$.

IV. The surface of a sphere is arbitrarily painted red and blue with 90% of the surface painted red. Prove that it is possible to inscribe a cube whose vertices are all red.

(a) Consider a randomly inscribed cube with vertices $v_1, \ldots, v_8$. Show that $\mathbb{P}[v_i \text{ is blue}] = 0.1$.
(b) Show that $\mathbb{P}[v_1 \text{ OR } v_2 \text{ OR } \cdots \text{ OR } v_8 \text{ is blue}] \le 0.8$.
(c) Hence show that $\mathbb{P}[\text{all eight vertices are red}] \ge 0.2$.
(d) What does it mean if a probability is positive?

V. Let $\mathbf{X}$ be the number of successes in n independent trials, each having probability of success p. Compute these probabilities:

(a) $\mathbb{P}[\text{at least one success}]$ and $\mathbb{P}[\text{at least one failure}]$.

(b) $\mathbb{P}$[even number of successes] and $\mathbb{P}$[even number of failures]. *Hint: use the Binomial Theorem to expand* $(x+y)^n - (y-x)^n$.

VI. You and a friend independently and repeatedly try to access a wireless channel, randomly with probability p at each step. The channel is accessible if one of you (not both) try to access. Let $\mathbf{X}$ be how long you wait for a first access and $\mathbf{Y}$ how long your friend waits. Give the PDF of: (a) $\mathbf{X}$ (b) $\mathbf{Y}$ (c) $\mathbf{Z} = \mathbf{X} - \mathbf{Y}$.

VII. Let the success probability in a trial be p. Let $\mathbf{X}$ be the waiting time for r successes. Derive the PDF of $\mathbf{X}$, i.e. compute $\mathbb{P}[\mathbf{X} = t]$.

(a) At which step is the rth success? In how many ways can you arrange the first $r-1$ successes?
(b) Show that $P_{\mathbf{X}}(t) = \binom{t-1}{r-1} p^r (1-p)^{t-r}$.
(c) Show that the above formula matches the PDF of the waiting time for one success when $r = 1$.

VIII. The number of fish in a lake is $\mathbf{X} \in [200, 400]$, with each value of $\mathbf{X}$ being equally likely. In a study, a biologist randomly caught 30 fish (without replacement), marked them, and replaced them in the lake. The next day, the biologist caught 30 random fish and found 20 to be marked.

(a) What was the PDF of $\mathbf{X}$ before the biologist did anything?
(b) What is the updated PDF of $\mathbf{X}$ after finding that 20 fish in the new sample are marked? (give a plot)
(c) What is the most likely number of fish in the lake?

Submission Problems

(1) You are a tourist in a foreign park, where there are twice as many tourists as locals. Locals hate the tourists and will always answer a question incorrectly. Tourists are random and answer repeated questions independently, giving the correct answer with probability $\frac{2}{3}$. You meet a random passerby and ask whether the exit is left or right.

(a) The answer is left. What is the probability the exit is left?
(b) You ask the same person and get left again. Now, what is the probability the exit is left?
(c) You ask a third time and get left again. Now, what is the probability the exit is left?
(d) You ask a fourth time. What are the chances the exit is left if: (a) the answer is left? (b) the answer is right?

(2) The manager of a movie theater announces that one free ticket will go to the first person in line whose birthday is the same as someone who has already bought a ticket. You can get into line at any time. You don't know anyone else's birthday, and birthdays are independent, being distributed randomly throughout the year (which has 365 days).

a) What position in line gives you the greatest chance of winning the free ticket?
b) What is the probability that you get the free ticket?

(3) Independently generate a 10-bit binary sequence $b_1 \cdots b_{10}$ with $\mathbb{P}[b_i = 0] = \frac{1}{2}$. Compute the probability that the sequence is sorted from low to high, e.g. 0000111111 is sorted.

(4) L people at a circular table with seats $0, 1, \ldots, L-1$ numbered counterclockwise pass the bread tray around. A person passes right with probability p and left with probability $1-p$, where $0 < p < 1$. The bread tray starts at seat 0. What is the probability P_k that the person seated at seat k is the last to receive the bread tray?

(5) Flip a fair coin n times. What is the probability of an equal number of H and T? Recompute the probability given the new information that the first flip is H.

(6) Let $\mathbf{X}$ be the number of successes in n independent trials and $\mathbf{Y}$ the number of successes in m additional trials (the probability of success in p in all trials). Let $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$. Give the PDF of $\mathbf{Z}$. Hence, show that if $\mathbf{X} \sim \text{Bin}(n, p)$ and $\mathbf{Y} \sim \text{Bin}(m, p)$ are independent, then $\mathbf{X} + \mathbf{Y} \sim \text{Bin}(n + m, p)$.

(7) Pick 10 fruits independently with probabilities: pear $\frac{1}{2}$, apple $\frac{1}{3}$, orange $\frac{1}{6}$. Compute the probabilities to get: (a) 5 pears. (b) 5 pears and 2 apples. (c) at least 5 pears or at least 2 apples.

(8) A lake has 600 fish. 60 have been marked by a biologist in a study. A year later, the biologist randomly caught (without replacement) 60 fish. Give the PDF for the number of marked fish in the second sample.