Name(s): ______________________________

## CSCI 2200 Foundations of Computer Science    Spring 2025

*Problem set 6 – Number theory; Graphs I*

**Instructions:** Standard course policies about typesetting, file size, and submission apply. You must show your work to receive credit. Your work must be your own, though you are permitted to get assistance from classmates or instructional staff. Your responses to the submission problems must be uploaded to Submitty by **8:59pm on Thursday, February 20**.
**Assume all graphs are simple graphs unless stated otherwise.**

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***Recitation Problems***

I. Prove that if $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

II. Calculate the number of zeros at the end of $1000!$. (That is "one thousand factorial" not an excited problem statement.)

III. Prove that $\gcd(F_{n+1}, F_n) = 1$, where $F_n$ is the $n$th Fibonacci number.

IV. What is the last digit of $3^{2025} + 4^{2025} + 7^{2025}$?

V. How many edges does each of these graphs have?
(a) $K_n$    (b) $K_{m,n}$    (c) $W_n$

VI. Given a graph $G$, the *complement* of $G$ is written as $\bar{G}$ and represents a new graph with the same vertices, but with precisely the edges that $G$ was missing: that is, if $(u, v)$ was an edge in $G$, then it is not an edge in $\bar{G}$, and vice versa. Prove that at least one of $G$ or $\bar{G}$ must be connected.

VII. A simple graph $G$ has $n$ vertices.
(a) What is the minimum number of edges $G$ could have and still be connected?
(b) What is the maximum number of edges $G$ could have and still **not** be connected?
Be sure to justify your answers.

VIII. Prove that, if every vertex in a simple graph $G$ has degree $d \geq 2$, then $G$ must contain a cycle of length at least $d + 1$.

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**Submission problems**

1. Compute the remainder when:
   (a) $2025^{2025}$ is divided by 7.  (b) $2600^{2600}$ is divided by 3.

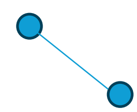
2. Prove that for all $k \in \mathbb{N}$, $2^k - 1$ and $2^k + 1$ are relatively prime. (Note: Not necessarily an induction proof.)

3. (a) When $n \in \mathbb{N}$, what are the possible values of $n^2\ (mod\ 4)$?
(b) Prove that none of 11, 111, 1111, 11111, ... are perfect squares.

4. Compute these modular multiplicative inverses, or explain why you cannot. Show your work.
   (a) $5^{-1}\ (mod\ 12)$  (b) $6^{-1}\ (mod\ 45)$  (c) $3^{-1}\ (mod\ 17)$

5. There is a group of 7 people. Six of the people in the group have exactly two friends within the group (obviously not the same pair of friends for each person). How many friends within the group can the 7th person have? Provide a graph illustrating each of your possibilities.

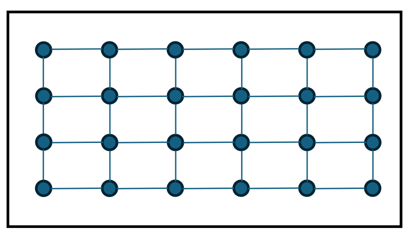
6. Claim: Every simple graph in which all vertices have degree > 0 is connected.
Proof: By induction on $n$, the number of vertices in the graph.
  Base case: n = 2. Each node has only one other node to connect to, and so in order to have degree > 0, it must be connected to the other node. (See right.)



  Induction step: Assume that the claim is true of all vertices with $n$ nodes where every node has degree > 0. If we add another node with degree > 0 to get to $n + 1$ nodes, it must connect to one of the existing nodes, and so the new graph is connected.

Give a counterexample to this claim, and explain where the flaw is in the inductive proof.

7. An *(r,c)-square grid graph* is one in which the vertices are laid out like the points with integer coordinates on the Cartesian plane; vertices are connected to the points that they are adjacent to horizontally or vertically. An example (4,6)-grid is shown at right.



(a) Count the number of vertices and edges in the (4,6)-grid.
(b) Give a formula in terms of r & c for the number of vertices and edges in an (r,c)-grid.
(c) Compute the degree distribution for an (r,c)-grid.

8. An arbitrary graph $G$ has precisely two vertices, $u$ and $v$, that have odd degree; every other vertex has even degree. Prove that there must be a path from $u$ to $v$ along the edges of $G$.