Name(s): ______________________________

CSCI 2200 Foundations of Computer Science Spring 2025

Problem set 5 – Structural induction; sums & asymptotics

Instructions: Standard course policies about typesetting, file size, and submission apply. You must show your work to receive credit. Your work must be your own, though you are permitted to get assistance from classmates or instructional staff. Your responses to the submission problems must be uploaded to Submitty by **8:59pm on Thursday, February 13**.

Reminder: At all times, $\log n$ refers to the **base-2** logarithm.

Recitation Problems

I. Prove via structural induction that the number of palindromes of length n is $2^{\lceil n/2 \rceil}$. (Note that $\lceil x \rceil$ is the *ceiling* of x, which is equal to the smallest integer greater than or equal to x.)

II. Matrix multiplication involves multiplying rows of the first matrix by columns of the second matrix. The values in the product matrix are the sums of these products. In the particular case of multiplying 2x2 matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Prove that $\forall n \in \mathbb{N}$, $A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$, where F_n is the nth Fibonacci number (assuming $F_0 = 0$ and $F_1 = 1$).

III. Find closed-form expressions for the following sums:

(a) $\sum_{i=1}^{n}(3i + 2i^2)$ (b) $\sum_{i=1}^{n}(-1)^i i$ (c) $\sum_{i=1}^{n}\sum_{j=1}^{m}(i + j)$

IV. Order the following function in order of increasing asymptotic growth. (That is, each one should be big-O of the next one in the list.

$n \log n$ $n!$ n^{100} $(1.5)^n$ n^n n $n\sqrt{n}$ 2^n $(\log n)^3$

V. Give the asymptotic big-Θ behavior of the runtime T_n:

(a) $T_0 = 1; T_n = T_{n-1} + n^2$

(b) $T_0 = 1; T_1 = 2; T_n = 2T_{n-1} - T_{n-2} + 2$

Submission problems

1. Consider the following recursively defined set:

$$3 \in A; \quad x, y \in A \rightarrow x + y \in A; \quad x, y \in A \rightarrow x - y \in A$$

(a) Prove that every element of A is a multiple of 3.
(b) Prove that every multiple of 3 is an element of A.

2. A *rooted ternary tree* is a tree structure in which each element has up to 3 children; a *rooted full ternary tree* is a rooted ternary tree in which each element has either 0 or 3 children.
(a) Write a recursive definition for a rooted full ternary tree. *(Hint:* Do not use the empty tree as a base case.)
(b) Prove via structural induction that the number of nodes in a rooted full ternary tree can always be written as $3k - 2$ for some $k \in \mathbb{N}$.

3. Find closed-form expressions for the following sums. <u>Show your work.</u>
(a) $\sum_{i=1}^{2n}(1 + 2i)$ (b) $\sum_{i=0}^{n} 2^{3+i}$ (c) $\sum_{i=0}^{n}\sum_{j=0}^{i} i2^j$

4. For each of the following, determine if they are $O(n^3)$, $\Omega(n^3)$, or $\Theta(n^3)$. <u>Show your work.</u>
(a) $n^2(\log n)^2$ (b) $n^3 + n^2$ (c) $n^{3.5}$ (d) $2^{2+3\log n}$ (e) $n^{\log n}$

5. Prove by contradiction that $n^3 \notin O(n^2)$.