Homework 10

Instructions: Standard course policies about typesetting, file size, and submission apply. You must show your work to receive credit. Your work must be your own, though you are permitted to get assistance from classmates or instructional staff. Your responses to the submission problems must be uploaded to Submitty by **8:59pm on Thursday, March 27**.

Attribution: these problems were chosen from the DMC text.

Note: you can employ Monte Carlo simulations to check some of your answers. But of course your arguments must be sound and not reference any simulations.

Recitation Problems

I. A random variable $\mathbf{X}$ takes values in $\{0, 1, 2 \ldots\}$. Show that

$$\mathbb{E}[\mathbf{X}] = \sum_{x=1}^{\infty} \mathbb{P}[\mathbf{X} \geq x] = \sum_{x=0}^{\infty} \mathbb{P}[\mathbf{X} > x].$$

This result is called the *tail sum formula* and is very useful.

II. A team is equally likely to win or lose its first game. In each following game, the previous result is twice as likely as the opposite result. What is the expected number of games played to get two wins?

III. A gambler walks into a casino with \$50 and plays roulette. The gambler bets \$1 on red (probability $\frac{18}{36}$) to win) and keeps betting until either going bankrupt or doubling his money. If it takes about 1 minute to play one game of roulette, how many hours of entertainment does the gambler expect to have?

IV. Pick random numbers from $\{1, \ldots, 100\}$ with replacement until their sum exceeds 100. How many numbers do you expect to pick?

V. A bag has 4 balls of different colors. At each step, pick two random balls and paint one ball the other ball's color. Replace the balls and repeat. On average, how many steps will it take until all balls are the same color?

VI. A fair coin is tossed repeatedly. Find the expected number of flips until HHH appears.

VII. Prove that every graph $G = (V, E)$ has a cut of size at least $|E|/2$. (A cut partitions V into two disjoint sets A, B; its size is the number of edges crossing from A to B).

(a) Construct the sets A, B by randomly placing vertices independently into one of the sets. Let $e = (u, v)$ be an edge in the graph. Compute $\mathbb{P}[u \text{ and } v \text{ are in different sets}]$.
(b) Define the indicator $\mathbf{X}(e) = 1$ if u and v are in different sets. Show that the value of the cut is $\mathbf{X} = \sum_{e \in E} \mathbf{X}(e)$. Compute $\mathbb{E}[\mathbf{X}]$ and prove the claim.

VIII. Let $\mathbf{X}$ be the wait for k successes with success probability p. Compute $\mathbb{E}[\max(0, r - \mathbf{X})]$ for $r \geq k$.

(a) Let $\mathbf{X}$ have pdf $P_{\mathbf{X}}(i)$ for $i \geq k$. Show that $\mathbb{E}[\max(0, r - \mathbf{X})] = \sum_{i=k}^{r} (r - i) \cdot P_{\mathbf{X}}(i)$.
(b) Establish that $P_{\mathbf{X}}(i) = \binom{i-1}{k-1} p^k (1-p)^{i-k}$.

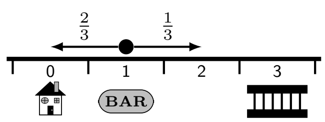
(c) Hence, show that

$$\mathbb{E}[\max(0, r - \mathbf{X})] = \begin{cases} r & k = 0; \\ \sum_{i=k}^{r} (r-i)\binom{i-1}{k-1} p^k (1-p)^{i-k} & k > 0 \end{cases}.$$

Submission Problems

(1) You get all heads in 10 flips of a random coin from a jar with 9 fair coins and 1 two-headed coin. Compute the expected number of heads in another 100 flips of the same coin.

(2) A drunk leaves the bar at position 1, and takes independent steps: left (L) with probability $\frac{2}{3}$ or right (R) with probability $\frac{1}{3}$. The drunk stops if he reaches home (at position 0) or the lockup (at position 3). Compute the expected number of steps for this scenario.



(3) What is the expected number of steps in the above problem, conditioned on the drunk making it home?

(4) A Martian couple has children until they have 2 males (sexes of children are independent). Compute the expected number of children the couple will have if, on Mars, males are: (a) Half as likely as females. (b) Just as likely as females. (c) Twice as likely as females.

(5) Use linearity and expectation and indicator random variables to answer these questions.

(a) You hash m items uniformly at random into n bins. Let $\mathbf{X}$ be the number of bins that contain exactly k items. What is $\mathbb{E}[\mathbf{X}]$?
(b) You randomly and independently choose a k-subset A and ℓ-subset B of $\{1, \ldots, n\}$. Let $\mathbf{X} = |A \cap B|$ and $\mathbf{Y} = |A \cup B|$. What are $\mathbb{E}[\mathbf{X}]$ and $\mathbb{E}[\mathbf{Y}]$?
(c) A street has n houses in a row that are painted randomly so that k are red and ℓ are white. Find the expected number of houses that have at least one neighbor that is a different color.

(6) A biased coin (probability p of heads) is tossed n times. A run is a consecutive sequence of the same outcome ($HHTHTTTTHH$ has five runs). Show that the expected number of runs is $1+2(n-1)p(1-p)$. Hint: focus on the transition between the runs.

(7) Fix $k > 1$ and suppose n is sufficiently small to satisfy $2\binom{n}{k}2^{-\frac{1}{2}k(k-1)} < 1$. Prove that some graph with n vertices has neither a k-clique nor a k-war. So, you need more than n vertices to guarantee a k-clique or a k-war.

Show this by constructing a random graph G on n vertices $v_1, \ldots, v_n$ by independently adding each edge (v_i, v_j) into the edge set with probability $\frac{1}{2}$, computing the expected total number of k-cliques and k-wars in G, and arguing that this result implies there is a graph on n vertices that has neither a k-clique nor a k-war.

(8) Let $\mathbf{X}$ be the waiting time to k boys and ℓ girls, assuming the probability of getting a boy in each birth is p. Let $\mathbf{X}_b$ be the time you wait for the k boys and let $\mathbf{X}_g$ be the *additional* time you must wait to get up to ℓ girls.

(a) What are the possible values of $\mathbf{X}_b$ and $\mathbf{X}_g$? Is $\mathbf{X}_g$ independent of $\mathbf{X}_b$? Explain.
(b) Show that $\mathbb{E}[\mathbf{X}_g \mid \mathbf{X}_b] = \max(0, k + \ell - \mathbf{X}_b)/(1 - p)$.
(c) Hence, show using iterated expectation that $\mathbb{E}[\mathbf{X}] = \frac{k}{p} + \frac{1}{1-p}\mathbb{E}[\max(0, k + \ell - \mathbf{X}_b)]$.
(d) Use the result from problem VIII of the recitation set to find an expression for $\mathbb{E}[\mathbf{X}]$.