# FOCS Exam 3 Spring 2025

**Instructions**: Legibly write your name and RIN below *and on your crib sheet*. Write as neatly as possible — we must be able to read your responses. Complete all problems to the best of your ability. Be sure to clearly explain your answers, especially in your proofs. You have until 10am to submit your exam.

No electronic devices of any type (including calculators) are permitted. The back of the exam is blank and may be used for scratch paper.

When you are done, bring your exam, crib sheet, and ID card to the front of the room. Wait until the proctor has confirmed that you are all set before leaving the room.

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Name: Grading Rubric

RCS userid: ____________________@rpi.edu

RIN: ______________________________

NB: – gave credit for two options on probs 3 and 7

**Multiple choice.** 10 points each. **Show your work for <u>all</u> answers** to receive full credit.

(1) A toddler bangs letters on a 26-letter keyboard that has lowercase letters only. Each letter is chosen independently and uniformly at random from the alphabet. If the toddler types 500,000 letters, what is the expected number of times the sequence "focs" appears?

A $\frac{499{,}997}{4} \cdot \frac{1}{26}$

B $499{,}997 \cdot \frac{1}{26^4}$

C $\sum_{k=1}^{500000} \binom{500000}{k} \left(\frac{1}{26}\right)^k \left(1 - \frac{1}{26}\right)^{500000-k}$

D $500{,}000 \cdot \frac{1}{26} \cdot \frac{1}{25} \cdot \frac{1}{24} \cdot \frac{1}{23}$

E None of the above

Using the indicator method,

$X = \sum_{i=1}^{499997} X_i \Rightarrow \mathbb{E}X = 499997 \mathbb{E}X_1$

$\Rightarrow \boxed{\mathbb{E}X = 499997 \frac{1}{26^4}}$

(2) A basketball player is practicing free throws. In the first round, they take 9 shots, and each shot goes in with a probability of 50%. If they make more shots than they miss, they get to take 9 more shots, which also each have a probability of 50% of going in. Let X be the total number of successful shots. What is $\mathbb{E}[X]$?

A $9 + \binom{9}{5}\frac{1}{2^9} \cdot 9$

B $9 + \left(1 - \binom{9}{5}\frac{1}{2^9}\right) \cdot 9$

C $\binom{9}{5}\frac{1}{2^9} \cdot 18 + \left(1 - \binom{9}{5}\frac{1}{2^9}\right) \cdot 9$

D $\binom{9}{5}\frac{1}{2^9} \cdot 9 + \left(1 - \binom{9}{5}\frac{1}{2^9}\right) \cdot 18$

E None of the above

$F \sim Bin(9, \frac{1}{2})$ is number shots that go in in the first round

$X = F + \begin{cases} 0 & \text{if } F < 5 \text{ shots} \\ F' & \text{otherwise} \end{cases}$, where $F'$ is i.i.d. with $F$

$\Rightarrow \mathbb{E}X = \mathbb{E}[F] + \mathbb{E}[F']\,\mathbb{P}(F \geq 5)$

$= \frac{9}{2}(1 + \mathbb{P}(F \geq 5))$ cont'd $\rightarrow$

(3) You roll two standard six-sided dice. Assuming the rolls are independent, find the probability that the product of the two dice is a perfect square.

A $\frac{1}{18}$

B $\frac{1}{12}$

C $\frac{5}{18}$

D $\frac{1}{6}$

E None of the above

$\mathbb{P}(XY \in \{1, 2^2, 3^2, 4^2, 5^2, 6^2\})$

$= \sum_{i=1}^{6} \mathbb{P}(X=i \cap Y=i) + \underbrace{\mathbb{P}(X=1 \cap Y=4) + \mathbb{P}(Y=1 \cap X=4)}$

$= \frac{8}{36} = \frac{2}{9}$

also count D as correct if left these cases out

2 cont'd)

We have $\mathbb{P}(F<4) + \mathbb{P}(F=5) + \mathbb{P}(F>5) = 1$ and by symmetry $(p=\frac{1}{2})$, the probabilities of sinking $<4$ shots and sinking $>5$ shots are identical, so

$$\mathbb{P}(F>5) = \frac{1}{2}\left[1 - \binom{9}{5}\frac{1}{2^9}\right]$$

and

$$\begin{aligned}\mathbb{P}(F \geqslant 5) &= \mathbb{P}(F=5) + \mathbb{P}(F>5)\\ &= \binom{9}{5}\frac{1}{2^9} + \frac{1}{2}\left[1 - \binom{9}{5}\frac{1}{2^9}\right]\\ &= \frac{1}{2}\left[1 + \binom{9}{5}\frac{1}{2^9}\right]\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbb{E}[F] &= \frac{9}{2}\left(\frac{3}{2} + \binom{9}{5}\frac{1}{2^9}\right)\\ &= \frac{27}{4} + 9 \cdot \binom{9}{5}\frac{1}{2^{10}}\end{aligned}$$

(4) A chef is preparing a mystery soup using a large pot containing 300 dumplings. 40 dumplings are spicy, while the rest are mild. The chef randomly scoops up 2 dumplings without replacement to taste. What is the probability that the dumplings are either both spicy or both mild?

- [A] $1 - \left(\frac{40}{300}\frac{260}{299} + \frac{260}{300}\frac{40}{299}\right)$
- [B] $1 - 2 \cdot \frac{40}{300}\frac{260}{299}$
- [C] $\frac{1}{2}(1 - \frac{40}{300}\frac{260}{299}) + \frac{1}{2}(1 - \frac{260}{300}\frac{40}{299})$
- [D] $\frac{40 \cdot 39 + 260 \cdot 259}{\binom{300}{2}}$
- [E] None of the above

$\mathbb{P}(\text{both spicy} \cup \text{both mild}) =$

$1 - \mathbb{P}(\text{one spicy} \cap \text{one mild}) =$

$1 - \left(\frac{40}{300} \cdot \frac{260}{299} + \frac{260}{300}\frac{40}{299}\right)$

or

$\mathbb{P}(\text{both spicy} \cup \text{both mild}) =$

$\mathbb{P}(\text{both spicy}) + \mathbb{P}(\text{both mild}) = \frac{40}{300} \cdot \frac{39}{299} + \frac{260}{300} \cdot \frac{259}{299}$

(5) The chances a telemarketer nets a sale on any given call is 40%. Compute the probability that more than 10 calls are required to make a sale.

- [A] $\left(\frac{3}{5}\right)^{10}$
- [B] $1 - \sum_{i=0}^{10} \binom{10}{i} \left(\frac{2}{5}\right)^i \left(\frac{3}{5}\right)^{10-i}$
- [C] $\frac{2}{3} \cdot \sum_{i=11}^{\infty} \left(\frac{3}{5}\right)^i$
- [D] $\sum_{i=10}^{\infty} \left(\frac{3}{5}\right)^i$
- [E] None of the above

$X =$ # calls for a sale

$\mathbb{P}(X > 10) = \mathbb{P}(\text{first 10 calls all failed})$

$= (1-p)^{10}$

$= (3/5)^{10}$

(6) You are a contestant on a game show where the host presents you with three coins. One coin is double-headed, one is double-tailed, and one is normal–with one heads and one tails. The host randomly selects a coin and flips it onto the table. You look down and see tails facing up. What is the probability that the other side is also tails?

- [A] $\frac{1}{3}$
- [B] $\frac{1}{2}$
- [C] $\frac{2}{3}$
- [D] $\frac{3}{4}$
- [E] None of the above

coin flip

$\frac{1}{3}$ HH $\xrightarrow{1}$ HH, H

$\frac{1}{3}$ TT $\xrightarrow{1}$ T, T

$\frac{1}{3}$ HT $\xrightarrow{1/2}$ HT, H

$\phantom{\frac{1}{3}}$ HT $\xrightarrow{1/2}$ TT, T

$\mathbb{P}(TT \mid T) =$

$\frac{\mathbb{P}(TT \,\&\, T)}{\mathbb{P}(T)}$

$= \mathbb{P}(TT) / \mathbb{P}(T)$

$= \frac{1}{3} / \left(\frac{1}{6} + \frac{1}{3}\right)$

$= 2/3$

(7) Let $P_n$ be the probability of at least two heads in $n$ fair coin tosses. Which of the following relations is true when $n > 2$?

$\boxed{\text{A}}$ $P_n = \frac{1}{2}P_{n-1} + \frac{1}{2}P_{n+1}$

$\boxed{\text{B}}$ $P_n = \frac{1}{2}P_{n-1} + \frac{1}{2} - (\frac{1}{2})^n$ also works $\longrightarrow$ see next page

$\boxed{\text{C}}$ $P_n = 1 - \frac{1}{2}P_{n-1}$

$\boxed{\text{D}}$ $P_n = \frac{1}{2}P_{n-1} + \frac{1}{2}P_1$

$\boxed{\text{E}}$ None of the above $P_n = P_{n-1} + (n-1)\frac{1}{2^n}$ works

(8) About 1 in 1000 people have a peanut allergy. The test for a peanut allergy makes a mistake on 2 in 10 people who have it (80% accuracy if you have a peanut allergy) and on 2 in 100 people who do not have it (98% accuracy if you do not have a peanut allergy). You got tested, and the result was positive. What is the probability that you have a peanut allergy?

$\boxed{\text{A}}$ $\dfrac{\frac{8}{10{,}000}}{\frac{8}{10{,}000}+\frac{2\cdot 999}{100{,}000}}$

$\boxed{\text{B}}$ $\dfrac{\frac{98}{100{,}000}}{\frac{98}{100{,}000}+\frac{8}{10{,}000}}$

$\boxed{\text{C}}$ $\dfrac{\frac{8}{10{,}000}}{\frac{8}{10{,}000}+\frac{2}{100{,}000}}$

$\boxed{\text{D}}$ $\dfrac{\frac{98}{100{,}000}}{\frac{98}{100{,}000}+\frac{2}{10{,}000}}$

$\boxed{\text{E}}$ None of the above

$$\mathbb{P}(\text{allergy} \mid \text{positive}) = \frac{\mathbb{P}(\text{positive} \mid \text{allergy})\mathbb{P}(\text{allergy})}{\mathbb{P}(\text{positive} \mid \text{allergy})\mathbb{P}(\text{allergy}) + \mathbb{P}(\text{positive} \mid \text{no allergy})\cdot\mathbb{P}(\text{no allergy})}$$

$$= \frac{8/10 \cdot 1/1000}{8/10 \cdot 1/1000 + 2/100 \cdot \frac{999}{1000}}$$

(9) You roll a fair die 40 times. What is the variance of the average of the rolls? (Recall that we computed the variance of the *sum of two* rolls in class: $\frac{35}{6}$.)

$\boxed{\text{A}}$ $\frac{35}{6}$

$\boxed{\text{B}}$ $\frac{35}{12}$

$\boxed{\text{C}}$ $40 \cdot \frac{35}{6}$

$\boxed{\text{D}}$ $40 \cdot \frac{35}{12}$

$\boxed{\text{E}}$ None of the above

$$Var(X+Y) = Var(X) + Var(Y) = 2Var(X) = \frac{35}{6}$$

$$\Rightarrow Var(X) = \frac{35}{12}$$

$$\Rightarrow Var(X_1 + \cdots + X_{40}) = 40 \cdot Var(X) = 40 \cdot \frac{35}{12}$$

$$Var(Avg(X_1 + \cdots + X_{40})) = \frac{1}{40^2} \cdot 40 \cdot \frac{35}{12} = \frac{1}{40} \cdot \frac{35}{12}$$

$\mathbb{P}(\geq 2 \text{ heads in } n \text{ tosses})$

$= \mathbb{P}(\geq 2$ heads in first $n-1$ tosses OR exactly one head in first $n-1$ tosses, then a head$)$

$$= P_{n-1} + \frac{1}{2} \cdot \binom{n-1}{1} \frac{1}{2}^{n-1}$$

$$= P_{n-1} + (n-1) \cdot \frac{1}{2^n}$$

OR

notice $P_n = 1 - \{p_0(n) + p_1(n)\}$ where

$p_0(n) =$ probability no heads in $n$ tosses

$p_1(n) =$ " one head in $n$ tosses

and further, by law of total expectation,

$$p_0(n) = \frac{1}{2} p_0(n-1) \Rightarrow p_0(n) = \frac{1}{2^n}$$

$$p_1(n) = \frac{1}{2} p_0(n-1) + \frac{1}{2} p_1(n-1)$$

so

$$p_n = 1 - \{p_0(n) + p_1(n)\} = 1 - [p_0(n-1) + \frac{1}{2} p_1(n-1)]$$

$$= 1 - \{\frac{1}{2}(p_0(n-1) + p_1(n-1)) + \frac{1}{2} p_0(n-1)\}$$

$$= \frac{1}{2} + \frac{1}{2}[1 - (p_0(n-1) + p_1(n-1))] - \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2} P_{n-1} - \frac{1}{2^n}$$

(10) Roll a pair of fair dice until the two dice sum to 7. What is the expected number of rolls?

A $\frac{7}{6}$

B 2

C 4

D 6

E None of the above

$X \sim$ waiting time to sum of 7

$\mathbb{P}(\text{one roll sums to } 7) =$

$\mathbb{P}((1,6) \text{ or } (2,5) \text{ or } (3,4) \text{ or } (4,3) \text{ or } (5,2) \text{ or } (6,1))$

$= \frac{1}{6}$

$\mathbb{E}X = \frac{1}{p} = 6$

**Short answer questions.** Respond to each prompt, showing all required work. **Write legibly!**

(11) [15 points] A total of $n$ bar magnets are placed end to end in a line with random independent orientations. Adjacent poles with the same polarities repel, while adjacent poles with opposite polarities join to form blocks. Let X be the number of blocks of joined magnets. Find $\mathbb{E}[X]$. Hint: use indicator random variables.

A new block starts at the ith position $(i = 2, \ldots, n)$ iff the polarities of the ith and (i-1)th blocks are the same, and position one always starts a block

$$\Rightarrow X = 1 + \sum_{i=2}^{n} X_i \quad \text{where } X_i = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \mathbb{E}X = 1 + (n-1)\mathbb{E}X_2 = 1 + \frac{(n-1)}{2}$$

$$= \frac{n+1}{2}$$

(12) [15 points] A plane has $n$ seats assigned to $n$ passengers, who randomly choose an available seat on boarding. What is the probability that the last passenger gets their assigned seat? Hint: a counting argument works.

$\mathbb{P}$(last passenger gets their seat) =

$\mathbb{P}$(first $n-1$ passengers do not sit in the $n^{th}$ passenger's seat)

$= \mathbb{P}$(passenger 1 doesn't sit in seat $n$)

$\mathbb{P}$(passenger 2 doesn't sit in seat $n$ | passenger 1 not sitting in seat $n$)

$\vdots$

$\mathbb{P}$(passenger $n-1$ doesn't sit in seat $n$ | passengers $1 \ldots n-2$ not sitting in seat $n$)

$$= \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdots \frac{n-(n-1)}{n-(n-2)} = \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdots \frac{2}{3} \cdot \frac{1}{2}$$

$$= \frac{1}{n}$$

OR

$\mathbb{P}$(last passenger gets their seat) =

$$\frac{\#\text{ permutations on } \{1, \ldots, n\} \text{ that fix } n}{\#\text{ permutations on } \{1, \ldots, n\}}$$

$$= \frac{(n-1)!}{n!} = \frac{1}{n}$$

(13) [20 points] Consider the set $\{1, ..., n\}$. We generate a subset $\mathcal{X}$ as follows: a fair coin is flipped independently for each element of the set; if the coin lands heads then the element is added to $X$, and otherwise it is not.

(a) Argue that the resulting set $\mathcal{X}$ is equally likely to be any one of the $2^n$ subsets.

(b) Suppose that two sets $\mathcal{X}$ and $\mathcal{Y}$ are chosen independently and uniformly at random from all the $2^n$ subsets of $\{1, \ldots, n\}$ in the manner given above. Determine $\mathbb{P}(\mathcal{X} \subseteq \mathcal{Y})$. Hint: $\mathbb{P}(\mathcal{X} \subseteq \mathcal{Y} \mid \mathcal{Y})$ only depends on $|\mathcal{Y}|$, the number of elements in $\mathcal{Y}$.

a) The subset $S$ is sampled iff the coin comes up heads for the elements of $S$ and comes up tails for the elements of $\bar{S}$. This occurs with probability $\frac{1}{2^{|S|}} \cdot \frac{1}{2^{|\bar{S}|}} = \frac{1}{2^n}$.

Thus $X$ is equally likely to be any of the $2^n$ subsets

b) $$\mathbb{P}(X \subseteq Y) = \sum_{S \subseteq \{1,\ldots,n\}} \mathbb{P}(X \subseteq Y \mid Y = S)\,\mathbb{P}(Y = S)$$

(using hint)

$$= \frac{1}{2^n} \sum_{S \subseteq \{1,\ldots,n\}} \mathbb{P}(X \subseteq Y \mid Y = S) = \frac{1}{2^n} \sum_{S \subseteq \{1,\ldots,n\}} 2^{|S|}/2^n$$

$$= \frac{1}{2^{2n}} \sum_{S \subseteq \{1,\ldots,n\}} 2^{|S|} = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{n}{k} 2^k$$

$\hookrightarrow$ OR

(simpler soln, from a TA)

b) Let $\mathbb{1}$ be a binary random vector of size $n$ chosen so

$$(\mathbb{1}_X)_i = \begin{cases} 1 & \text{if } i \in X \\ 0 & \text{if } i \notin X \end{cases} \sim \text{Bin}\left(\tfrac{1}{2}\right)$$

and define $\mathbb{1}_Y$ similarly.

Note the entries of $\mathbb{1}_X$ and $\mathbb{1}_Y$ are independent, so

$$\mathbb{P}(X \subseteq Y) = \mathbb{P}(\forall i : (\mathbb{1}_X)_i \leq (\mathbb{1}_Y)_i)$$

$$= \prod_{i=1}^{n} \mathbb{P}((\mathbb{1}_X)_i \leq (\mathbb{1}_Y)_i) = \mathbb{P}(z \leq z')^n$$

where $z$ and $z'$ are i.i.d. $\text{Bin}\left(\frac{1}{2}\right)$ r.v.s.

Note $\mathbb{P}(z \leq z') = \mathbb{P}(z=0 \;\&\; z'=0 \text{ OR } z=0 \;\&\; z'=1 \text{ OR } z=1 \;\&\; z'=1)$

$$= \frac{3}{4}$$

so

$$\mathbb{P}(X \subseteq Y) = \left(\frac{3}{4}\right)^n$$