CSCI 2200 Foundations of Computer Science – Spring 2025 – Exam 2

Name: ___Solution Key____

RCS userid: ___________________________________@rpi.edu

RIN: ____

General instructions: *<u>Legibly</u> write your name and RIN above* ***<u>and on your crib sheet</u>****. Write as neatly as possible – we must be able to read your responses. Complete all problems to the best of your ability. Be sure to <u>clearly</u> explain your answers, especially in your proofs. You have until 10am to submit your exam.*

No electronic devices of any type (including calculators) are permitted. The back of the exam is blank and may be used for scratch paper.

When you are done, bring your exam, crib sheet, and ID card to the front of the room. Wait until the proctor has confirmed that you are all set before leaving the room.

Part 1 – Multiple choice (5 points each)

Circle the letter for the ***<u>single</u>*** *most appropriate answer. When indicated with a ♦, you must* ***<u>show work</u>*** *in the space to the right of the choices or you will not receive credit even if your answer is correct.*

1. How many natural numbers less than 55 are relatively prime to 55? ♦Show your work.

A. 14

B. 15

C. 40

D. 54

E. None of the above

Since 55 = 5 × 11, count multiples of 5 and multiples of 11. There are 10 multiples of 5 from 1 to 54, and 4 multiples of 11 from 1 to 54, so 14 total natural numbers less than 55 that are NOT relatively prime to 55. Thus, there are 54 – 14 = 40 natural numbers less than 55 that are relatively prime to 55.

2. How many ways are there to select a president, vice-president, secretary, and treasurer from a club with 20 members?

A. ${}_{20}P_4$ The four positions are distinguishable, making this a permutations problem.

B. ${}_{23}P_3$

C. $\binom{20}{4}$

D. $\binom{23}{3}$

E. None of the above

3. Compute $\sum_{i=1}^{2n}(1+3i)$. (*Please note the sum goes to* $\underline{2n}$.) ♦Show your work.

A. $3n^2+5n$

B. $6n^2+5n$

C. $6n^2+8n$

D. $12n^2+8n$

E. None of the above

$\sum_{i=1}^{2n}(1+3i)=\sum_{i=1}^{2n}1+3\sum_{i=1}^{2n}i$ by the Sum and Constant rules. This leaves two common sums, using $2n$ in place of n, giving $2n+3\left(\frac{1}{2}(2n)(2n+1)\right)$. Simplifying, we get $2n+3(2n^2+2n)=6n^2+8n$.

4. Let $T(1)=1, T(2)=2$, and $T(n)=2T(n-1)-5T(n-2), \forall n\geq 3$. What is $T(6)$? ♦Show your work.

A. -1

B. 1

C. 12

D. 22

E. None of the above

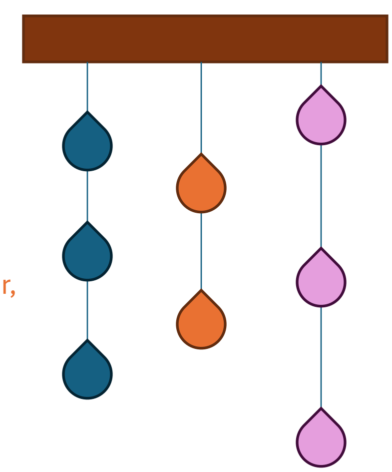
$T(3)=2(2)-5(1)=-1$

$T(4)=2(-1)-5(2)=-12$

$T(5)=2(-12)-5(-1)=-19$

$T(6)=2(-19)-5(-12)=22$

5. A carnival game involves a set of eight hanging water balloons: three in the first column, two in the second, and three in the third. (See diagram at right.) Players throw darts, and they win if they break all of the balloons, but there is a catch: they must always break the *lowest* remaining balloon in each column. In how many different orders can a player break the balloon while following the rules and thus win the game?



A. $8!$

B. $8!\div 3!$

C. $8!\div(3!\cdot 2!\cdot 3!)$

D. $10!\div(8!\cdot 2!)$

E. None of the above

Consider: The player must select Left, Center, or Right each time. How many ways can you rearrange the letters LLLCCRRR?

Part 2 – Multiple selection (7 points each)

Circle the letter of each answer that is correct; ***if there are two or more, you must select all that apply****.*

When indicated with a ♦, you must ***show work*** *or you will not receive credit even if your answer is correct.*

6. Which of the following situations would be correctly counted using the expression $\binom{15}{3}$?

A. The number of 15-bit strings that have exactly 3 zeros

B. The number of ways to select a dozen donuts at a store that sells four varieties (glazed, frosted, apple cider, and Boston cream)

C. At a county fair pie-baking contest with 15 entrants, the number of ways to award a blue, a red, and a white ribbon

D. The number of different three-person teams that can be selected from a class of 15 students

E. None of the above

7. Which of the following are possible chromatic numbers (minimum # of colors needed) for W_n, the wheel graph on n nodes ($n \geq 4$)? ♦ Explain your answer (which could involve drawing graph(s)).

A. 2

B. 3

C. 4

D. 5

A wheel graph is a cycle graph with a hub node. A cycle graph with an even number of vertices requires two colors (you can alternate around the cycle), while a cycle with an odd number of vertices requires three colors. Since the hub node connects to all other vertices, it must get its own color, adding one to the overall total.

8. ♦ For this question, give a brief explanation for each choice.
Which of the following functions are $O(n^2)$?

A. $100\ n \log n$ — $\forall \boldsymbol{n} > \mathbf{1}, \log n < n$. Therefore, $100\ n \log n < \mathbf{100}\ \boldsymbol{n}$^**2**

B. $12n^2 + 98n + 16$ — $\forall \boldsymbol{n} > \mathbf{1}, 12n^2 + 98n + 16 < 12n^2 + 98n^2 + 16n^2 \leq \mathbf{126}\boldsymbol{n}^{\mathbf{2}}$

C. $6n \cdot 1.5^n$ — All exponentials with base > 1 grow faster than all polynomials.

D. $3n^2\sqrt{n}$ — This is just $3n^{2.5}$, and $n^{2.5}$ grows faster than n^2.

9. Which of the following are equivalent to 1 (mod 11)? ♦ Show work / explain why for each choice.

A. 25 $25 \div 11 = 2R3$

B. -10 $-10 + 11 = 1$

C. 3^{10} 11 is prime. By Euler's Little Theorem, $a^{11-1} \equiv 1\ (mod\ 11)$ for all $a \neq 0$.

D. 11^{10} 11 mod 11 is 0, so $11^{10} \equiv 0\ (mod\ 11)$.

10. Consider the degree sequence [3,3,3,3,1,1]. Which kind of simple graph could it describe? ♦ Explain your answer (which could involve drawing graph(s)).

A. None – it's an impossible sequence

B. A disconnected graph

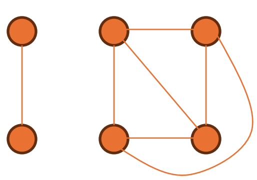
C. A connected tree

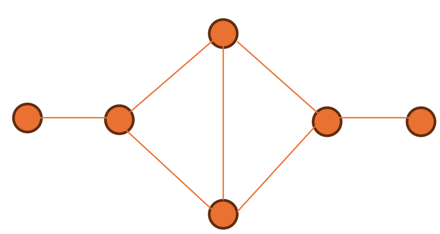
D. A connected planar graph

E. A connected non-planar graph

F. A complete graph

There are only two possible simple graphs with this degree sequence, shown below.





Part 3 – Free response

Respond to each prompt, showing all required work. ***Write legibly***!

11. A *regular* graph is one in which every vertex has the same non-zero degree. Consider a regular bipartite graph, where the two sets of vertices are X and Y.

[9 points] (a) Prove that $|X| = |Y|$.

[6 points] (b) Prove that the graph must contain a perfect matching.

(a) Let d be the degree of every vertex in the graph, and let $n = |X|$. Then the sum of the degrees of all of the vertices in X must be nd. Since the graph is bipartite, every edge must have one vertex in X and one vertex in Y. Therefore, the sum of the degrees of all of the vertices in Y is also nd. Since each of these vertices also has degree d, there must be n of them and $|Y| = n$. ■

(b) Again, let d be the degree of every vertex in the graph. Assume, for contradiction, that there is no perfect matching. Then, by Hall's Theorem, there must be a subset of vertices S on one side of the matching with strictly fewer neighbors than $|S|$. Let $a = |S|$ and $b = |n(S)|$, and so $a > b$. Because the graph is regular, the total number of edges leading out of S is ad; all of these edges must lead into $n(S)$ [That's what "neighbors" means.], and so $n(S)$ must have at least ad edges. But because the graph is regular, $n(S)$ has exactly bd edges, and $bd < ad$. Contradiction! Therefore our assumption was false, and there is a perfect matching. ■

12. [10 points] Let P be a set of points in the Cartesian plane, defined recursively as follows:

$$(1,0) \in P \quad \text{and} \quad (x, y) \in P \rightarrow (x+1, y+2) \in P$$

Prove using structural induction that $\forall (x, y) \in P, y = 2x - 2$.

Base case: $(1,0)$ satisfies the equation: $0 = 2(1) - 2$

Inductive claim: If (x, y) satisfies $y = 2x - 2$, then $(x + 1, y + 2)$ satisfies $(y + 2) = 2(x + 1) - 2$.

Using the I.H., assume that $y = 2x - 2$.

Add 2 to both sides but DON'T cancel the 2s, obtaining: $y + 2 = 2x + 2 - 2$.

Finally, factor out a 2 on the right, reaching: $y + 2 = 2(x + 1) - 2$, which is the goal. ■

13. [15 points] How many positive integers less than 10,000 contain the digit 3? *(There are several ways to approach this problem. It is acceptable to leave your answer in the form of an arithmetic expression without performing the final calculations, as long as you show your work.)*

Here is one possible approach: Break it down into cases with exactly one 3, exactly two 3s, exactly three 3s, and exactly four 3s. Then, since they're disjoint, add these up at the end using the Sum Rule.

- Exactly one 3: There are $\binom{4}{1}$ possible locations for the 3. Once that is fixed, the other three digits can be anything else (and their order matters), so there are 9^3 possibilities for those. So there are $\binom{4}{1} \cdot 9^3$ natural numbers below 10,000 with exactly one 3.
- Exactly two 3s: In similar fashion, there are $\binom{4}{2}$ options for the 3s and 9^2 possibilities for the other digits, giving $\binom{4}{2} \cdot 9^2$.
- Exactly three 3s: $\binom{4}{3} \cdot 9^1$
- Exactly four 3s: $\binom{4}{4}$

So in total there are $\binom{4}{1} \cdot 9^3 + \binom{4}{2} \cdot 9^2 + \binom{4}{3} \cdot 9 + \binom{4}{4}$ natural numbers below 10,000 that contain the digit 3.

(You weren't required to finish the calculation, but that comes out to $4 \cdot 729 + 6 \cdot 81 + 4 \cdot 9 + 1 = 3{,}439$)