Foundations of Computer Science
Lecture 10

Number Theory

Division and the Greatest Common Divisor
Fundamental Theorem of Arithmetic
Cryptography and Modular Arithmetic
RSA: Public Key Cryptography

Last Time

1. Why sums and reccurrences? Running times of programs.
2. Tools for summation: constant rule, sum rule, common sums and nested sum rule.
3. Comparing functions - asymptotics: Big-Oh, Theta, Little-Oh notation.
$$\log\log(n) < \log^{\alpha}(n) < n^{\epsilon} < 2^{\delta n}$$
4. The method of integration - estimating sums.

$$\sum_{i=1}^{n} i^k \sim \frac{n^{k+1}}{k+1} \qquad \sum_{i=1}^{n} \tfrac{1}{i} \sim \ln n \qquad \ln n! = \sum_{i=1}^{n} \ln i \sim n\ln n - n$$

Today: Number Theory

1. Division and Greatest Common Divisor (GCD)
   - Euclid's algorithm
   - Bezout's identity
2. Fundamental Theorem of Arithmetic
3. Modular Arithmetic
   - Cryptography
   - RSA public key cryptography

The Basics

Number theory has attracted the best of the best, because

"Babies can ask questions which grown-ups can't solve" – P. Erdős

$6 = 1 + 2 + 3$ is *perfect* (equals the sum of its proper divisors). Is there an odd perfect number?

Quotient-Remainder Theorem

For $n \in \mathbb{Z}$ and $d \in \mathbb{N}$, $n = qd + r$. The quotient $q \in \mathbb{Z}$ and remainder $0 \le r < d$ are *unique*.

e.g. $n = 27, d = 6$: $\quad 27 = 4 \cdot 6 + 4 \quad \rightarrow \quad \text{rem}(27, 6) = 4$.

Divisibility. d divides n, $d|n$ if and only if $n = qd$ for some $q \in \mathbb{Z}$. e.g. $6|24$.

Primes. $\text{P} = \{2, 3, 5, 7, 11, \ldots\} = \{p \mid p \ge 2$ and the only positive divisors of p are $1, p\}$.

Division Facts (Exercise 10.2)

1. $d|0$.
2. If $d|m$ and $d'|n$, then $dd'|mn$.
3. If $d|m$ and $m|n$, then $d|n$.
4. If $d|n$ and $d|m$, then $d|n + m$.
5. If $d|n$, then $xd|xn$ for $x \in \mathbb{N}$.
6. If $d|m + n$ and $d|m$, then $d|n$.

Greatest Common Divisor

Divisors of 30: $\{1, 2, 3, 5, 6, 15, 30\}$. Divisors of 42: $\{1, 2, 3, 6, 7, 14, 21, 42\}$. Common divisors: $\{1, 2, 3, 6\}$.

$$\textit{greatest common divisor (GCD)} = 6.$$

Definition. Greatest Common Divisor, GCD

Let m, n be two integers not both zero. $\gcd(m, n)$ is the largest integer that divides both m and n: $\gcd(m, n)|m$, $\gcd(m, n)|n$ and any other common divisor $d \leq \gcd(m, n)$.

Notice that every common divisor divides the GCD. Also, $\gcd(m, n) = \gcd(n, m)$.

Relatively Prime

If $\gcd(m, n) = 1$, then m, n are relatively prime.

Example: 6 and 35 are not prime but they are relatively prime.

Theorem.

$\gcd(m, n) = \gcd(\text{rem}(n, m), m)$.

Proof. $n = qm + r \to r = n - qm$. Let $D = \gcd(m, n)$ and $d = \gcd(m, r)$.
$D|m$ and $D|n \to D$ divides $r = n - qm$. Hence, $D \leq \gcd(m, r) = d$. (D is a common divisor of m, r)
$d|m$ and $d|r \to d$ divides $n = qm + r$. Hence, $d \leq \gcd(m, n) = D$. (d is a common divisor of m, n)
$D \leq d$ and $D \geq d \to D = d$, which proves $\gcd(m, n) = \gcd(n, r)$. ■

Euclid's Algorithm

Theorem.
$\gcd(m, n) = \gcd(\text{rem}(n, m), m)$.

$$\begin{aligned}
\gcd(42, 108) &= \gcd(24, 42) && 24 = \mathbf{108} - 2 \cdot \mathbf{42} \\
&= \gcd(18, 24) && 18 = 42 - 24 = 42 - \underbrace{(108 - 2 \cdot 42)}_{24} = 3 \cdot \mathbf{42} - \mathbf{108} \\
&= \gcd(6, 18) && 6 = 24 - 18 = \underbrace{(108 - 2 \cdot 42)}_{24} - \underbrace{(3 \cdot 42 - 108)}_{18} = 2 \cdot \mathbf{108} - 5 \cdot \mathbf{42} \\
&= \gcd(0, 6) && 0 = 18 - 3 \cdot 6 \\
&= 6 && \gcd(0, n) = n
\end{aligned}$$

Remainders in Euclid's algorithm are integer linear combinations of 42 and 108.

In particular, $\gcd(42, 108) = 6 = 2 \times 108 - 5 \times 42$.

This will be true for $\gcd(m, n)$ in general:

$$\gcd(m, n) = mx + ny \qquad \text{for some } x, y \in \mathbb{Z}.$$

Bezout's Identity: A "Formula" for GCD

From Euclid's Algorithm,

$$\gcd(m,n) = mx + ny \qquad \text{for some } x, y \in \mathbb{Z}.$$

Can any smaller positive number z be a linear combination of m and n?

$$\text{suppose:} \qquad z = mx + ny > 0.$$

$\gcd(m,n)$ divides RHS $\rightarrow \gcd(m,n)|z$, i.e $z \geq \gcd(m,n)$ (because $\gcd(m,n)|m$ and $\gcd(m,n)|n$).

Theorem. Bezout's Identity

$\gcd(m,n)$ is the *smallest positive integer linear combination* of m and n:

$$\gcd(m,n) = mx + ny \qquad \text{for } x, y \in \mathbb{Z}.$$

Formal Proof. Let ℓ be the smallest positive linear combination of m, n: $\ell = mx + ny$.

- Prove $\ell \geq \gcd(m,n)$ as above.
- Prove $\ell \leq \gcd(m,n)$ by showing ℓ is a common divisor ($\text{rem}(m,\ell) = \text{rem}(n,\ell) = 0$).

There is no "formula" for GCD. But this is close to a "formula".

GCD Facts

- (i) $\gcd(m,n) = \gcd(m, \text{rem}(n,m))$. ✓
- (ii) Every common divisor of m, n divides $\gcd(m,n)$. ✓
- (iii) For $k \in \mathbb{N}$, $\gcd(km, kn) = k \cdot \gcd(m,n)$. ✓
- (iv) IF $\gcd(l,m) = 1$ AND $\gcd(l,n) = 1$, THEN $\gcd(l, mn) = 1$. ✓
- (v) IF $d|mn$ AND $\gcd(d,m) = 1$, THEN $d|n$. ✓

Proof.

- (ii) $\gcd(m,n) = mx + ny$. Any common divisor divides the RHS and so also the LHS.

  (e.g. 1,2,3,6 are common divisors of 30,42 and all divide the GCD 6)

- (iii) $\gcd(km, kn) = kmx + kny = k(mx + ny)$. The RHS is the smallest possible, so there is no smaller positive linear combination of m, n. That is $\gcd(m,n) = (mx + ny)$.

  (e.g. $\gcd(6,15) = 3 \to \gcd(12,30) = 2 \times 3 = 6$)

- (iv) $1 = \ell x + my$ and $1 = \ell x' + ny'$. Multiplying,

$$1 = (\ell x + my)(\ell x' + ny') = \ell \cdot (\ell x x' + nxy' + myx') + mn \cdot (yy').$$

  (e.g. $\gcd(15,4) = 1$ and $\gcd(15,7) = 1 \to \gcd(15,28) = 1$)

- (v) $dx + my = 1 \to ndx + nmy = n$. Since $d|mn$, d divides the LHS, hence $d|n$, the RHS.

  (e.g. $\gcd(4,15) = 1$ and $4|15 \times 16 \to 4|16$)

∎

Given 3 and 5-gallon jugs, measure exactly 4 gallons.

1: Repeatedly fill the 3-gallon jug.

2: Empty the 3-gallon jug into the 5-gallon jug.

3: If ever the 5-gallon jug is full, empty it by discarding the water.

$$(0,0)\xrightarrow{1:}(3,0)\xrightarrow{2:}(0,3)\xrightarrow{1:}(3,3)\xrightarrow{2:}(1,5)\xrightarrow{3:}(1,0)\xrightarrow{2:}(0,1)\xrightarrow{1:}(3,1)\xrightarrow{2:}\mathbf{(0,4)}\checkmark$$

After the 3-gallon jug is emptied into the 5-gallon jug, the state is $(0,\ell)$, where

$$\ell = 3x - 5y.$$

(the 3-gallon jug has been emptied x times and the 5-gallon jug y times)

(integer linear combination of $3,5$). Since $\gcd(3,5) = 1$ we can get $\ell = 1$,

$$1 = 3 \cdot 2 - 5 \cdot 1$$

(after emptying the 3-gallon jug 2 times and the 5 gallon jug once, there is 1 gallon)

Do this 4 times and you have 4 gallons (guaranteed). (Actually fewer pours works.)

$$(0,0)\xrightarrow{1:}(3,0)\xrightarrow{2:}(0,3)\xrightarrow{1:}(3,3)\xrightarrow{2:}(1,5)\xrightarrow{3:}(1,0)\xrightarrow{2:}(0,1) \qquad \text{(repeat 4 times)}$$

If the producers of Die Hard had chosen 3 and 6 gallon jugs, there can be no sequel (phew ☺). (Why?)

Fundamental Theorem of Arithmetic Part (ii)

Theorem. Uniqueness of Prime Factorization

Every $n \geq 2$ is *uniquely* (up to reordering) a product of primes.

Euclid's Lemma: For primes $p, q_1, \ldots, q_\ell$, if $p|q_1 q_2 \cdots q_\ell$ then p is one of the q_i.

Proof of lemma: If $p|q_\ell$ then $p = q_\ell$. If not, $\gcd(p, q_\ell) = 1$ and $p|q_1 \cdots q_{\ell-1}$ by GCD fact (v). Induction on ℓ.

Proof. (FTA) Contradiction. Let n_* be the smallest counter-example, $n_* > 2$ and

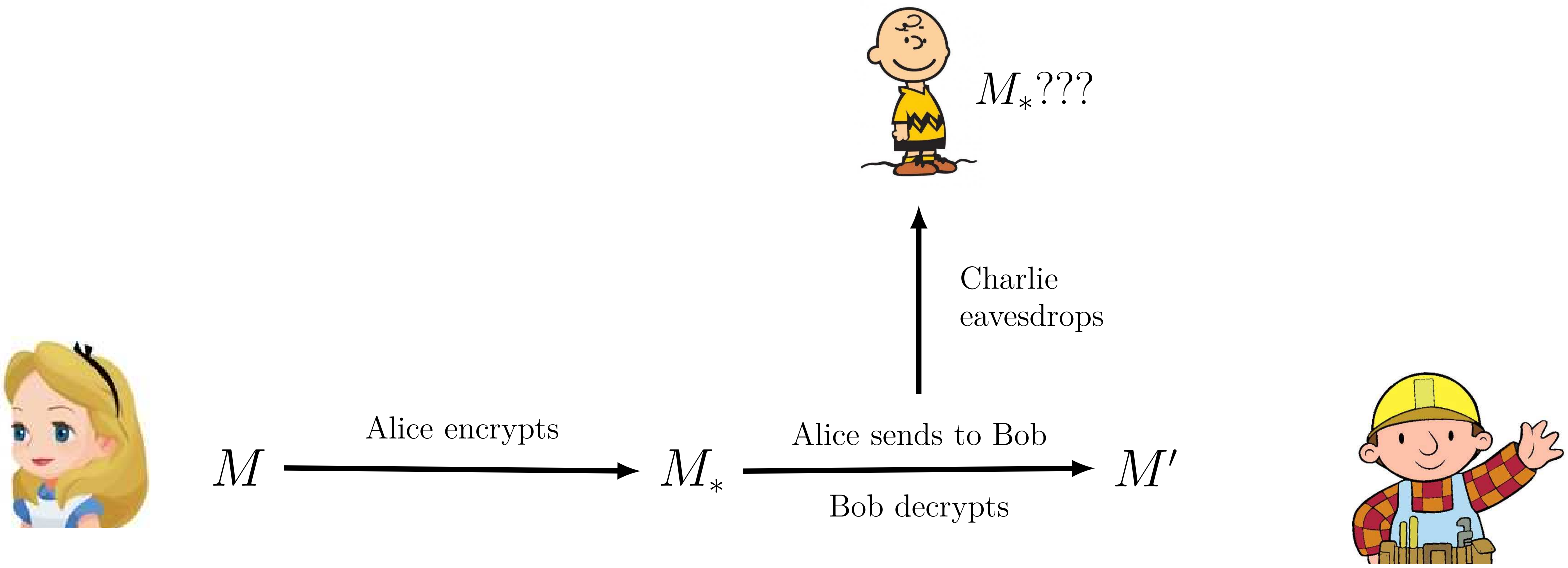
$$\begin{aligned} n_* &= p_1 p_2 \cdots p_n \\ &= q_1 q_2 \cdots q_k \end{aligned}$$

Since $p_1|n_*$, it means $p_1|q_1 q_2 \cdots q_k$ and by Euclid's Lemma, $p_1 = q_i$ (w.l.o.g. q_1).

$$\begin{aligned} n_*/p_1 &= p_2 \cdots p_n \\ &= q_2 \cdots q_k. \end{aligned}$$

That is, n_*/p_1 is a smaller counter-example. **FISHY!** ∎

Cryptography 101: Alice and Bob wish to securely exchange the prime M



Example.

Alice Encrypts: $M_* = M \times k$ (k is a shared secret – *private key*)

Alice and Bob know k, Charlie does not.

Bob Decrypts: $M' = M_*/k = M \times k/k = M$. (Hooray, $M' = M$ and Charlie is in the dark.)

Secure as long as Charlie cannot factor M' into k and M. (Factoring is HARD)

One time use. For two *cypher-texts*, $k = \gcd(M_{1*}, M_{2*})$.

To improve, we need modular arithmetic.

Modular Arithmetic

$$a \equiv b \pmod{d} \qquad \text{if and only if} \qquad d|(a-b), \qquad \text{i.e. } a-b = kd \text{ for } k \in \mathbb{Z}$$

$$41 \equiv 79 \pmod{19} \qquad \text{because} \qquad 41 - 79 = -38 = -2 \cdot 19.$$

Modular Equivalence Properties.

Suppose $a \equiv b \pmod{d}$, i.e. $a = b + kd$, and $r \equiv s \pmod{d}$, i.e. $r = s + \ell d$. Then,

(a) $ar \equiv bs \pmod{d}$.

$$\begin{aligned} & ar - bs \\ = \; & (b+kd)(s+\ell d) - bs \\ = \; & d(ks + b\ell l + k\ell d). \end{aligned}$$

That is $d|ar - bs$.

(b) $a + r \equiv b + s \pmod{d}$.

$$\begin{aligned} & (a+r) - (b+s) \\ = \; & (b+kd+s+\ell d) - b - s \\ = \; & d(k+\ell). \end{aligned}$$

That is $d|(a+r)-(b+s)$.

(c) $a^n \equiv b^n \pmod{d}$.

Repeated application of (a)
Induction.

Addition and multiplication are just like regular arithmetic.

Example. What is the last digit of 3^{2017}?

$$\begin{aligned} 3^2 &\equiv -1 \pmod{10} \\ \to \quad (3^2)^{1008} &\equiv (-1)^{1008} \pmod{10} \\ \to \; 3 \cdot (3^2)^{1008} &\equiv 3 \cdot (-1)^{1008} \pmod{10} \\ &\equiv 3 \end{aligned}$$

Modular Division is Not Like Regular Arithmetic

$$15 \cdot \cancel{6} \equiv 13 \cdot \cancel{6} \pmod{12} \qquad 15 \cdot \cancel{6} \equiv 2 \cdot \cancel{6} \pmod{13} \qquad 7 \cdot \cancel{8} \equiv 22 \cdot \cancel{8} \pmod{15}$$

$$15 \not\equiv 13 \pmod{12} \ ✘ \qquad 15 \equiv 2 \pmod{13} \ ✓ \qquad 7 \equiv 22 \pmod{15} \ ✓$$

Modular Division: cancelling a factor from both sides

Suppose $ac \equiv bc \pmod{d}$. You can cancel c to get $a \equiv b \pmod{d}$ if $\gcd(c, d) = 1$.

Proof. $d|c(a-b)$. By GCD fact (v), $d|a-b$ because $\gcd(c, d) = 1$. ■

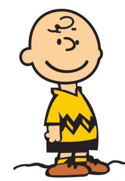
If d is prime, then division with prime modulus is pretty much like regular division.

Modular Inverse. Inverses do not exist in $\mathbb{N}$. Modular inverse may exist.

$$\begin{aligned} 3 \times n &= 1 & & n = ? \\ 3 \times n &= 1 \pmod{7} & & n = 5 \end{aligned}$$

RSA Public Key Cryptography Uses Modular Arithmetic

Bob broadcasts to the world the numbers 23, 55. (Bob's RSA *public key*).



M_*???

Charlie eavesdrops

$$M \xrightarrow{\text{Alice encrypts}} M_* \equiv M^{23} \pmod{55} \xrightarrow[\text{Bob decrypts}]{\text{Alice sends to Bob}} M' \equiv M_*^{7} \pmod{55}$$

Examples. Does Bob always decode to the correct message?

$M = 2.$ $M_* = 8$ $M' = 2$ $M' = M$

$2^{23} \equiv 8 \pmod{55}$ $8^7 \equiv 2 \pmod{55}$

$M = 3.$ $M_* = 27$ $M' = 3$ $M' = M$

$3^{23} \equiv 27 \pmod{55}$ $27^7 \equiv 3 \pmod{55}$

Exercise 10.14. Proof that Bob always decodes to the right message for special 55,23 and 7. (How to get them?)

Practical Implementation. Good idea to pad with random bits to make the cypher text random.