Foundations of Computer Science
Lecture 8

Proofs with Recursive Objects

Structural Induction: Induction on Recursively Defined Objects
Proving an object is *not* in a recursive set
Examples: sets, sequences, trees

Last Time

1. Recursion.
2. Recurrences are recursive functions on $\mathbb{N}$.
3. Recursive programs.
4. Recursive sets.
5. Rooted binary trees (RBT).

Today: Proofs with Recursive Objects

1. Two Types of Questions About Recursive Sets

2. Matched Parentheses

3. Structural Induction
   - $\mathbb{N}$
   - Palindromes
   - Arithmetic Expressions

4. Rooted Binary Trees (RBT)

Two Types of Questions About a Recursive Set

$\mathcal{A} = \{0,$

Recursive definition of $\mathcal{A}$.

1. $0 \in \mathcal{A}$.

Two Types of Questions About a Recursive Set

$$\mathcal{A} = \{0, 4,$$

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2. $x \in \mathcal{A} \rightarrow x + 4 \in \mathcal{A}$.

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(i) What is in $\mathcal{A}$? Is some feature common to every element of $\mathcal{A}$? Is everything in $\mathcal{A}$ even?

$$x \in \mathcal{A} \;\to\; x \text{ is even} \qquad \text{(T)}$$

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(ii) Is everything with some property in $\mathcal{A}$? Is every even number in $\mathcal{A}$?

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Very very different statements!

Every leopard has 4 legs.
Everything with 4 legs is a leopard?

Structural induction shows every member of a recursive set has a property, question (i).

Orks and blue Eyes

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When could a green-eyed ork have arisen?

Structural Induction

- The ancestors have a trait.
- The trait is passed on from parents to children.

Conclusion: Everyone today has that trait.

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The strings in $\mathcal{M}$ are the matched (in the arithmetic sense) parentheses. For example:

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\begin{array}{ll}
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Let's list the strings in $\mathcal{M}$ as they are created,

$$\mathcal{M} = \{\ \underset{s_1}{\underset{\uparrow}{\varepsilon}},\ \underset{s_2}{\underset{\uparrow}{[\,]}},\ \underset{s_3}{\underset{\uparrow}{[[\,]]}},\ \underset{s_4}{\underset{\uparrow}{[\,][\,]}},\ \underset{s_5}{\underset{\uparrow}{[[\,]][\,]}},\ \ldots,\ s_n,\ \ldots\ \}$$

To get s_n, we apply the constructor to two prior (not necessarily distinct) strings.

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(Orks inherit blue eyes. Here, parents pass along balance to the children.)

Just as all Orks will have blue eyes, all strings in $\mathcal{M}$ will be balanced.

Proof: Strings in $\mathcal{M}$ are Balanced

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Proof. Strong induction on n.

1: **[Base case]** The base case is $s_1 = \varepsilon$ which is clearly balanced, so $P(1)$ is T.

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2: **[Induction step]** Show $P(1) \wedge \cdots \wedge P(n) \to P(n+1)$ (direct proof).
Assume $P(1) \wedge P(2) \wedge \cdots \wedge P(n)$*:* $s_1, \ldots, s_n$ are all balanced.

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Therefore s_{n+1} is balanced (because you add one opening and closing parenthesis).

3: By induction, $P(n)$ is T $\forall n \geq 1$. ■

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Question. Is every balanced string in $\mathcal{M}$?

Exercise. Prove that $[[] \notin \mathcal{M}$.

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3: By structural induction, conclude that $P(s)$ is T for all $s \in \mathcal{S}$. ■

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- **MUST** show for *every* base case.

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- **MUST** show for *every* base case.
- **MUST** show for *every* constructor rule.
- Structural induction can be used with *any* recursive set.

Every String in $\mathcal{M}$ is Matched

$$[\ [\ ]\ ]\ [\ ]$$

opening:3 closing:3

Going from left to right:

Every String in $\mathcal{M}$ is Matched

$$[\ [\ ]\ ]\ [\ ]$$

opening:3 closing:3

Going from left to right:

$$[$$

opening:1 closing:0

Every String in $\mathcal{M}$ is Matched

$$[\ [\ ]\ ]\ [\ ]$$

opening:3 closing:3

Going from left to right:

$$[\ [$$

opening:2 closing:0

Every String in $\mathcal{M}$ is Matched

$$[\ [\ ]\ ]\ [\ ]$$

opening:3 closing:3

Going from left to right:

$$[\ [\ ]$$

opening:2 closing:1

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opening:3 closing:3

Going from left to right:

$$[\ [\ ]\ ]$$

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opening:3 closing:2

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$$[\ [\ ]\ ]\ [\ ]$$

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Every String in $\mathcal{M}$ is Matched

$$[\;[\;]\;]\;[\;]$$

opening:3 closing:3

Going from left to right:

$$[\;[\;]\;]\;[\;]$$

opening:3 closing:3

Opening is always at least closing: parentheses are arithmetically matched.

Important Exercise. Prove this by structural induction.

Key step is to show that constructor preserves "matchedness".

Every String in $\mathcal{M}$ is Matched

$$[\ [\ ]\ ]\ [\ ]$$

opening:3 closing:3

Going from left to right:

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Opening is always at least closing: parentheses are arithmetically matched.

Important Exercise. Prove this by structural induction.

Key step is to show that constructor preserves "matchedness".

Question. Is every string of matched parentheses in $\mathcal{M}$?

Hard Exercise. Prove this. (see Exercise 8.3).

Structural Induction on $\mathbb{N}$

$\mathbb{N} = \{1, 2, 3, \ldots\}$ is a recursively defined set,

1. $1 \in \mathbb{N}$.
2. $x \in \mathbb{N} \to x + 1 \in \mathbb{N}$.

Structural Induction on $\mathbb{N}$

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That's just ordinary induction! ☺

Palindromes $\mathcal{P}$

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Therefore, we can prove by structural induction that all strings in $\mathcal{P}$ are palindromes.

Hard Exercise. Prove that all palindromes are in $\mathcal{P}$ (Exercise 8.7).

Arithmetic Expressions

Fact known to all kindergartners: $((1+1+1)\times(1+1+1+1+1)) = 15$,

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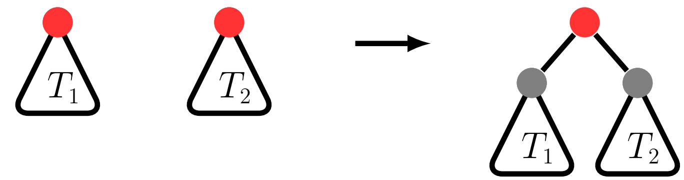
Constructors preserve "oddness" $\to$ all strings in $\mathcal{A}_{\text{ODD}}$ have odd value.

Rooted Binary Tree with $n \geq 1$ Vertices Have $n - 1$ Edges

1. The empty tree ε is an RBT.
2. Disjoint RBTs T_1, T_2 give a new RBT by linking their roots to a new root.



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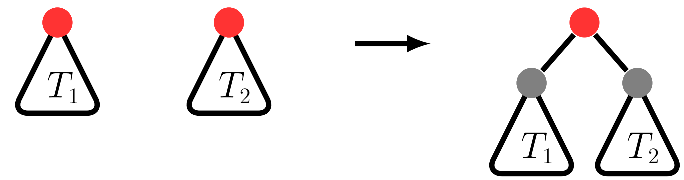

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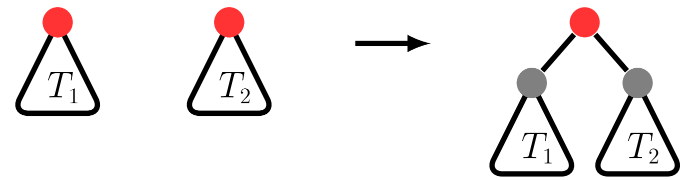

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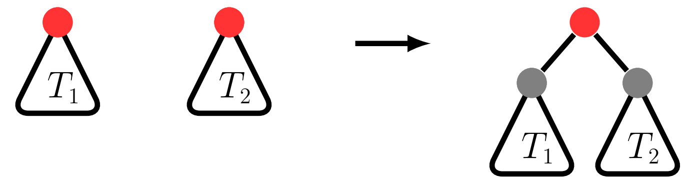

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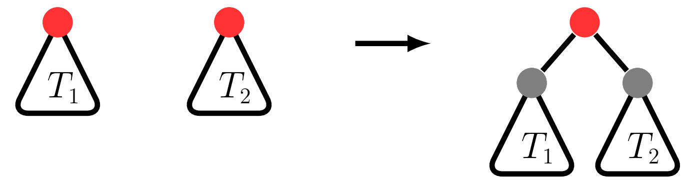

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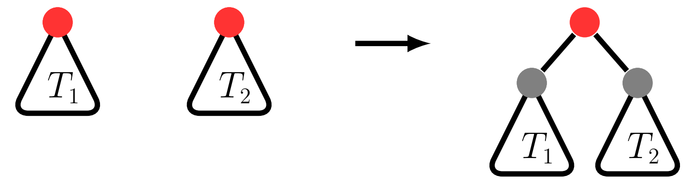

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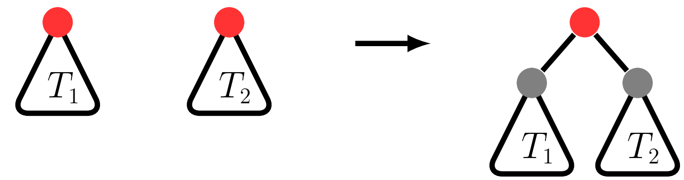

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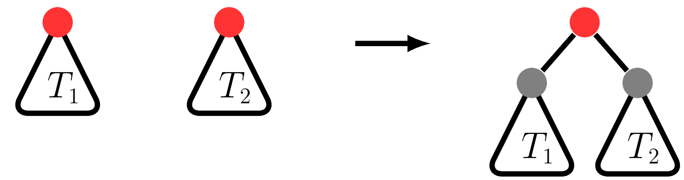

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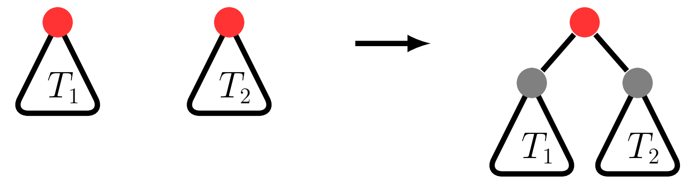

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Constructor always preserves property P.

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Analogy: if the first ancestors had blue eyes, and blue eyes are inherited from one generation to the next, then all of society will have blue eyes.

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Checklist for Structural Induction

Analogy: if the first ancestors had blue eyes, and blue eyes are inherited from one generation to the next, then all of society will have blue eyes.

- ✓ You have a recursively defined set $\mathcal{S}$.
- ✓ You want to prove a property P for all members of $\mathcal{S}$.
- ✓ Does the property P hold for the base cases?
- ✓ Is the property P *preserved* by all the constructor rules?
- ✗ Structural induction is not how to prove all objects with property P are in $\mathcal{S}$.