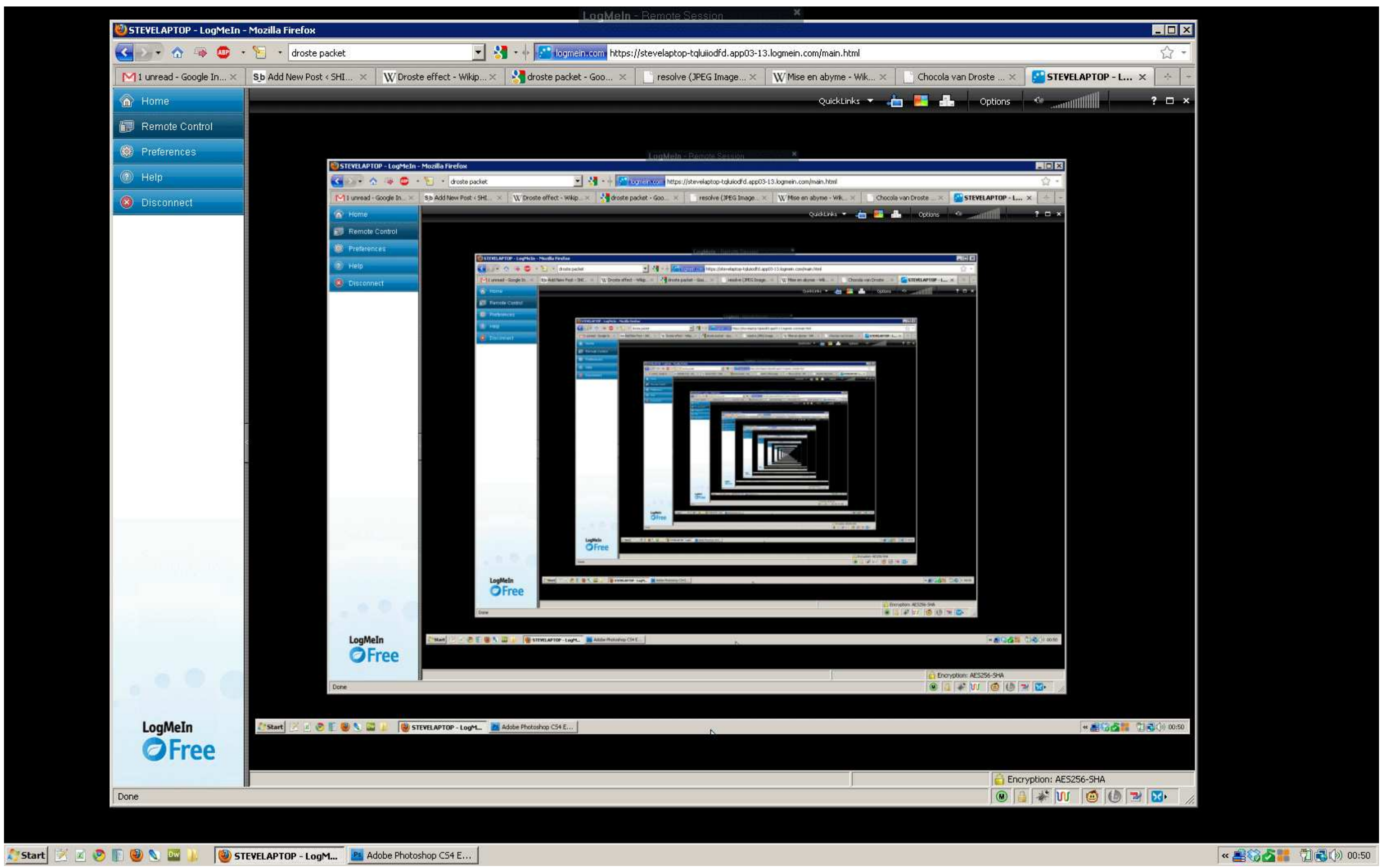
Foundations of Computer Science
Lecture 7

Recursion

Powerful but Dangerous
Recursion and Induction
Recursive Sets and Structures



Last Time

1. With induction, it may be easier to prove a stronger claim.
2. Leaping induction.
   - $n^3 < 2^n$ for $n \geq 10$.
   - Postage.
3. Strong induction.
   - Representation theorems: **FTA**, binary expansion.
   - Games: Nim with 2 equal piles.

Today: Recursion

1. Recursive functions
   - Analysis using induction
   - Recurrences
   - Recursive programs
2. Recursive sets
   - Formal Definition of $\mathbb{N}$
   - The Finite Binary Strings Σ^*
3. Recursive structures
   - Rooted binary trees (RBT)

A Fantastic Recursion

Online lecture tool “Demo”: allows lecturer to see screen of remote student.

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PROFESSOR

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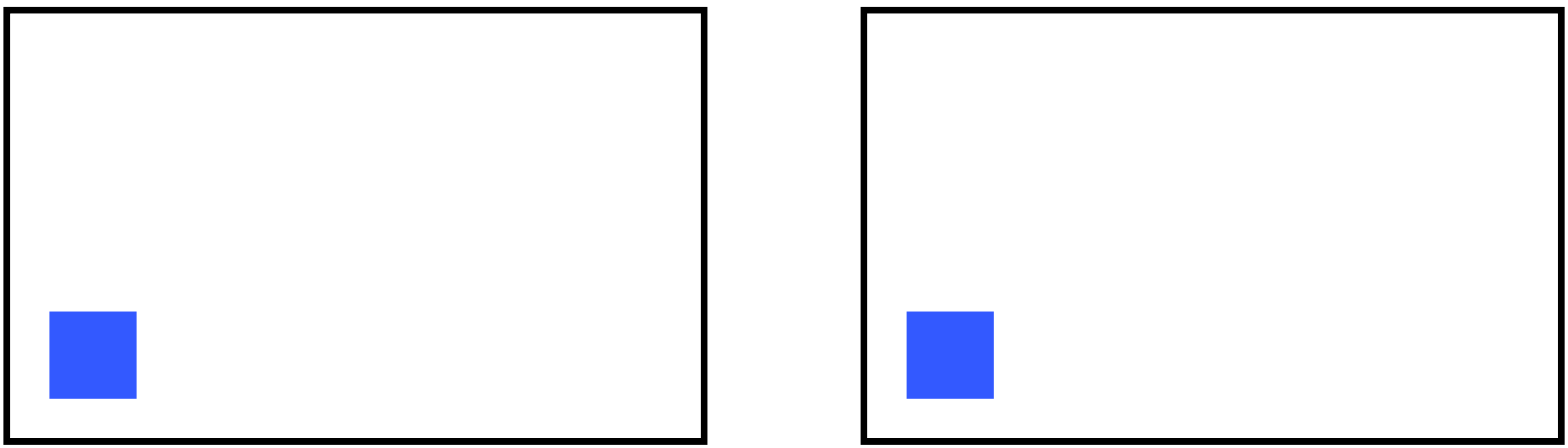
PROFESSOR STUDENT



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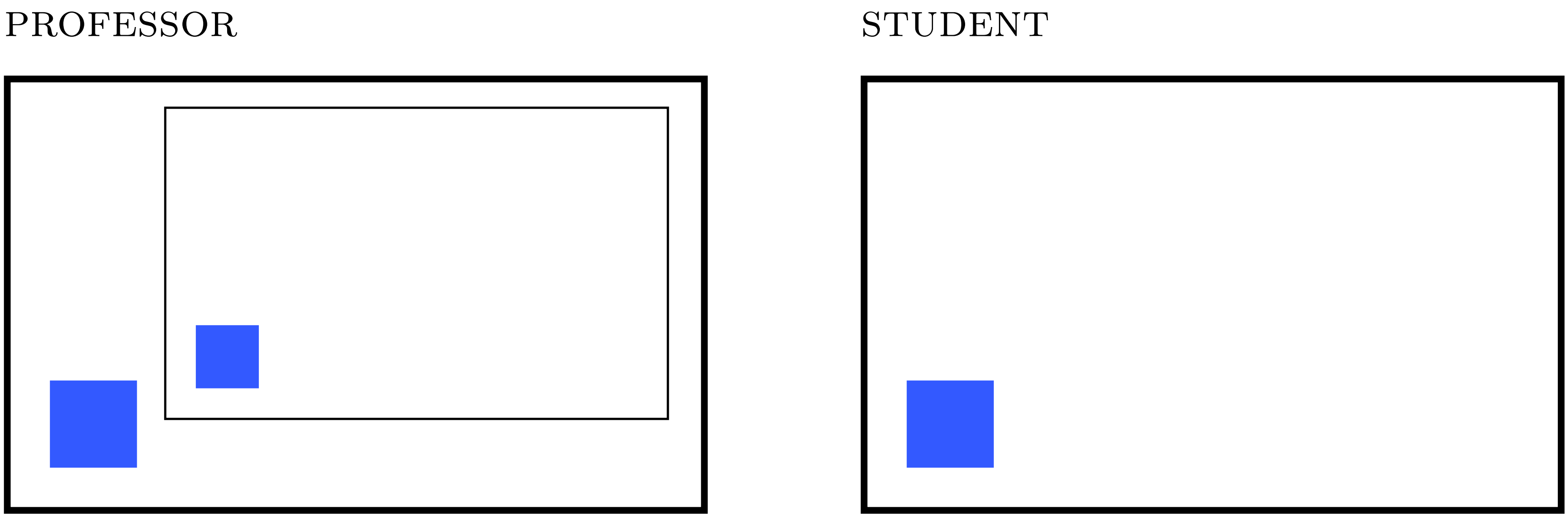

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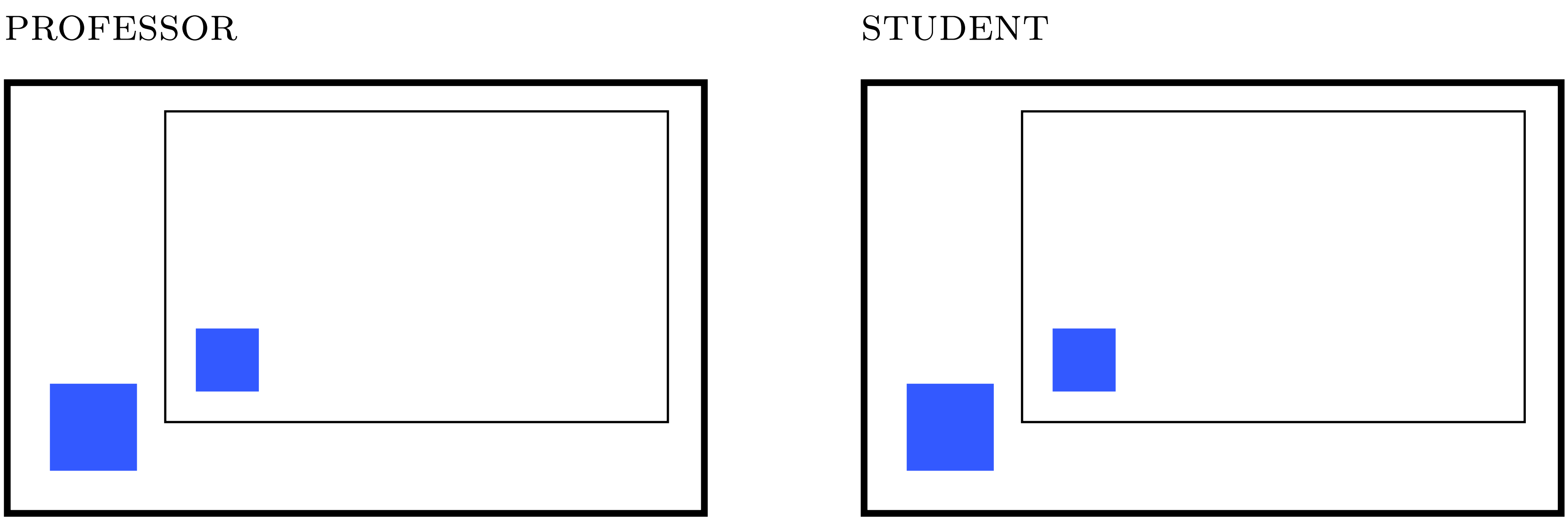

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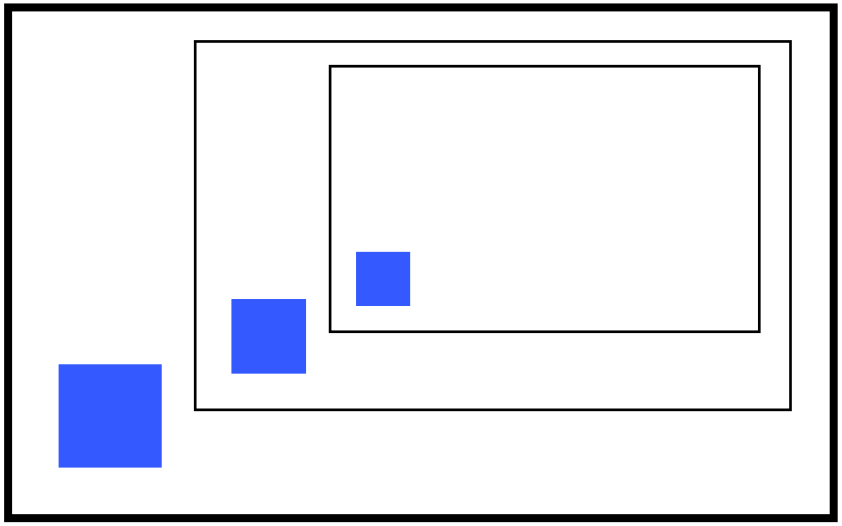

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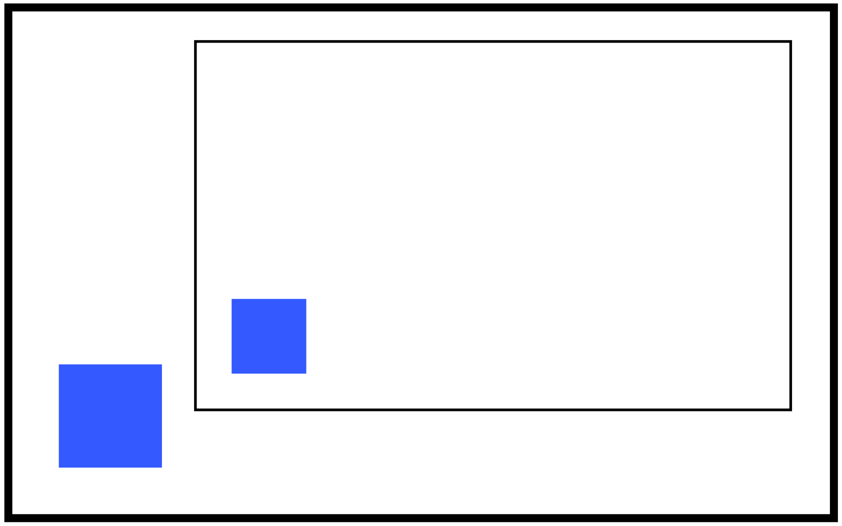

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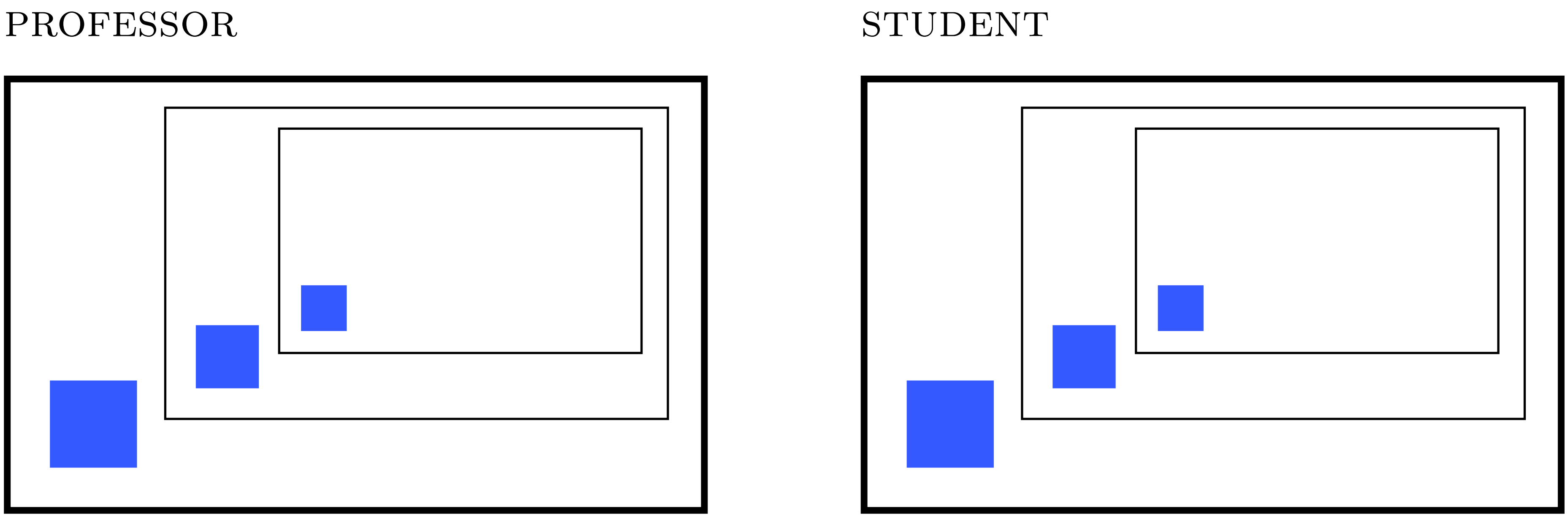

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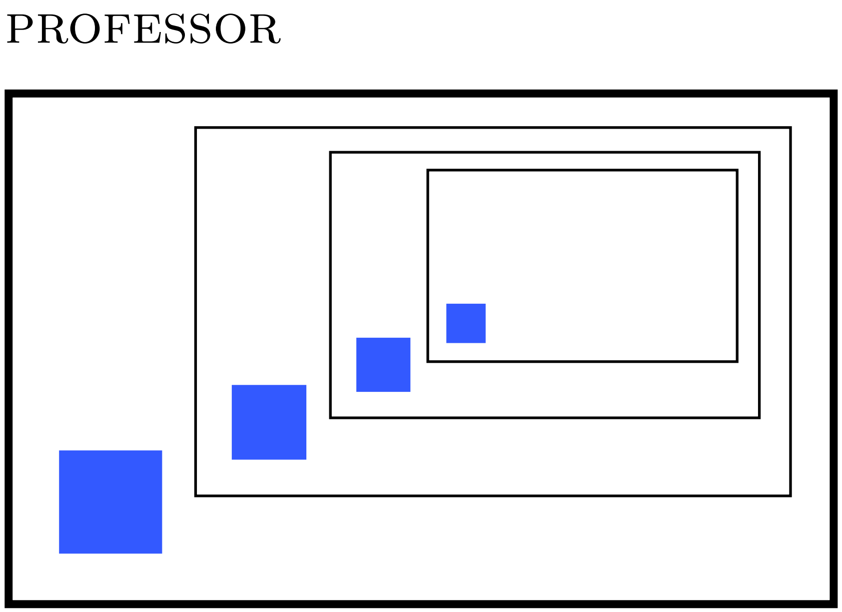

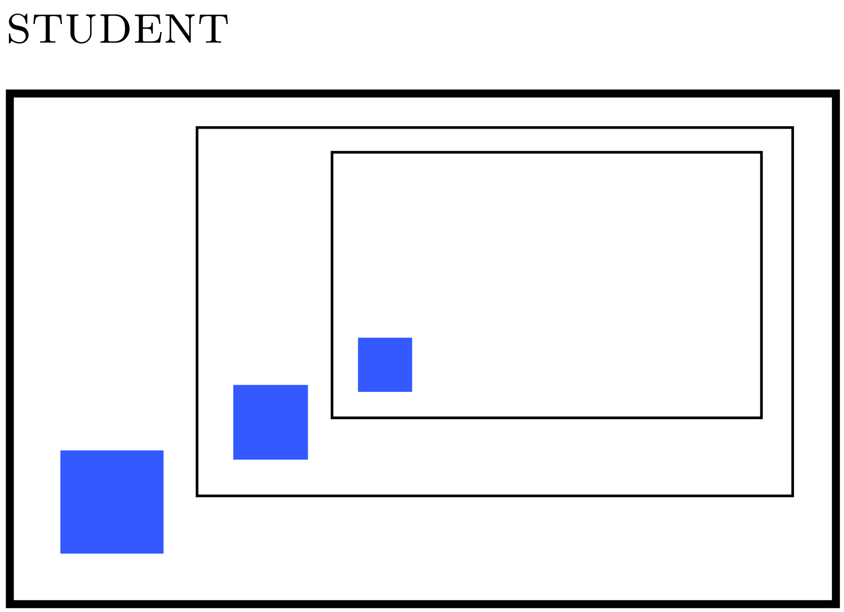

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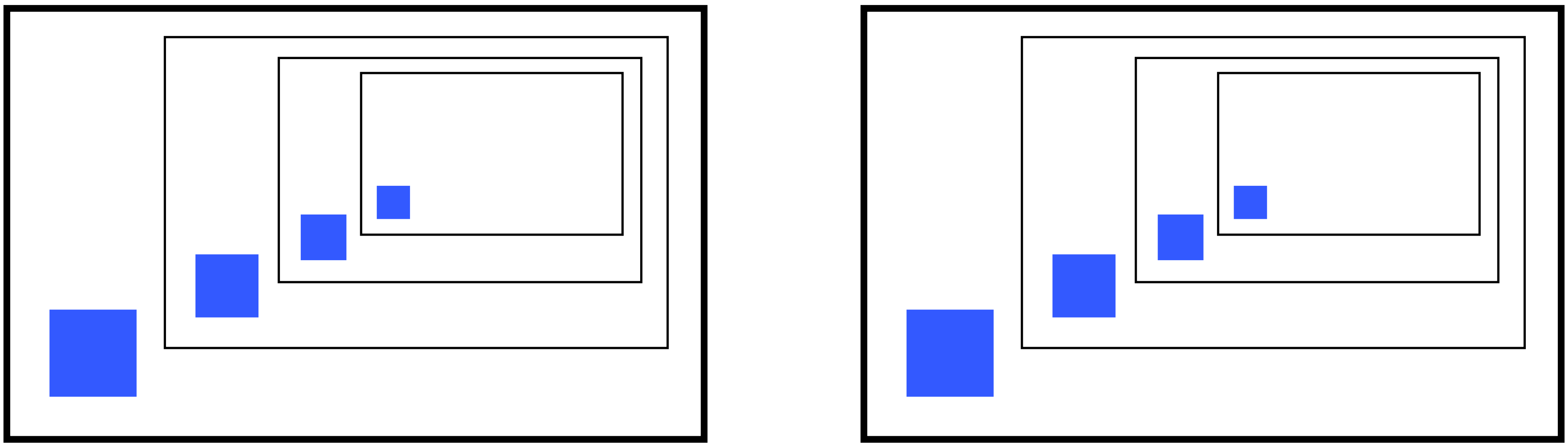

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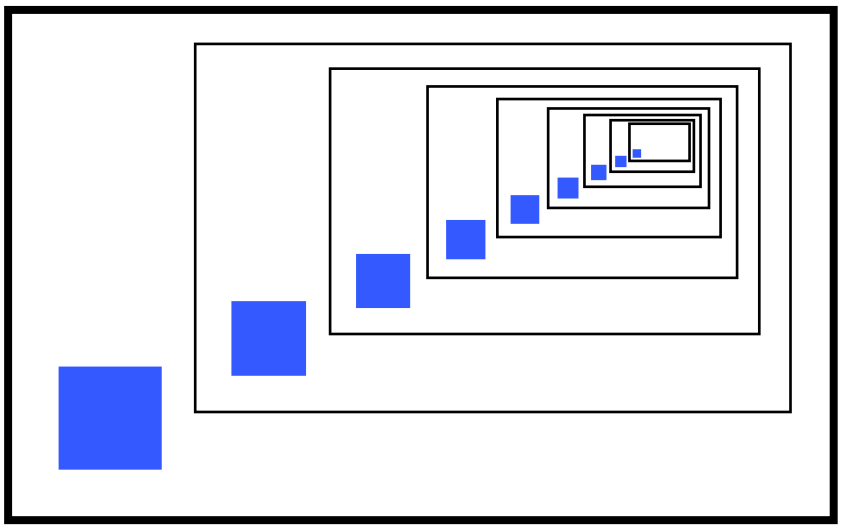

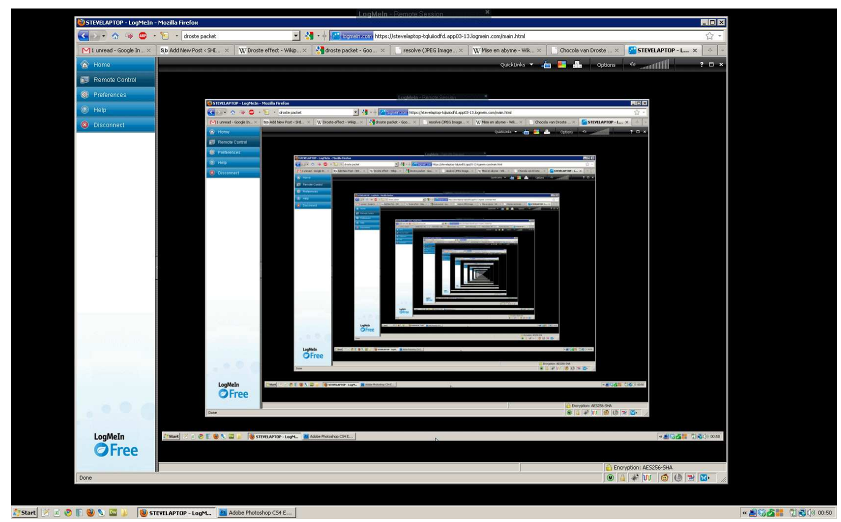

HANG!, CRASH!, BANG!, reboot required */?%&# ☹@$#!

Examples of Recursion: Self Reference

The tool shows the student's screen, i.e my previous screen, which is what the tool showed,

The tool *shows* what the tool *showed*. – *self reference*

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***/?%&# ☹@$#!**

Recursion Must Have Base Cases: *Partial* Self Reference.

look-up (word) works if there are some known words to which everything reduces.

Similarly with recursive functions,

$$f(n) = \begin{cases} 0 & n \leq 0; \\ f(n-1) + 2n - 1 & n > 0. \end{cases}$$

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Must have **base cases:**
In this case $f(0)$.

Must make **recursive progress:**
To compute $f(n)$ you must move *closer* to the base case $f(0)$.

Recursion and Induction

$$f(n) = \begin{cases} 0 & n \le 0; \\ f(n-1) + 2n - 1 & n > 0. \end{cases}$$

$\boxed{\text{f(0)}}$

Recursion and Induction

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Induction

$P(0)$ is T; $P(n) \to P(n+1)$

(you can conclude $P(n+1)$ if $P(n)$ is T)

$$\boxed{P(0)} \to P(1) \to P(2) \to P(3) \to P(4) \to \cdots$$

$P(n)$ is T for all $n \geq 0$.

Recursion

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Recursion

$f(0) = 0$; $f(\boldsymbol{n+1}) = f(n) + 2n + 1$

(we can *compute* $f(n+1)$ if $f(n)$ is known)

$\boxed{f(0)} \to f(1) \to f(2) \to f(3) \to f(4) \to \cdots$

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We can compute $f(n)$ for all $n \ge 0$.

Example: More Base Cases

$$f(n) = \begin{cases} 1 & n = 0; \\ f(n-2) + 2 & n > 0. \end{cases}$$

n	0	1	2	3	4	5	6	7	8
$f(n)$	1	✘							

Recursion and Induction

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How to fix $f(n)$? Hint: leaping induction.

$\boxed{\boldsymbol{f(0)}}$ $f(1)$ $f(2)$ $f(3)$ $f(4)$ $f(5)$ $f(6)$ $f(7)$ $f(8)$ $\cdots$

Practice. Exercise 7.4

Using Induction to Analyze a Recursion

$$f(n) = \begin{cases} 0 & n \le 0; \\ f(n-1) + 2n - 1 & n > 0. \end{cases}$$

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$f(n)$	0	1	4	9	16	25	36	49	64	$\cdots$

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Unfolding the Recursion

$$f(n) = f(n-1) + \; 2n - 1$$

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$$\begin{aligned} f(n) &= f(n-1) + \ 2n - 1 \\ f(n-1) &= f(n-2) + \ 2n - 3 \end{aligned}$$

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$$\begin{aligned} f(n) &= f(n-1) + \ 2n-1 \\ f(n-1) &= f(n-2) + \ 2n-3 \\ f(n-2) &= f(n-3) + \ 2n-5 \end{aligned}$$

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$+$

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&\vdots \\
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\cancel{f(1)} &= \cancel{f(0)}^{\,0} + 1 \\
\hline
+ &
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Proof by induction that $f(n) = n^2$.

$P(n) : f(n) = n^2$

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[Base case] $P(0) : f(0) = 0^2$ (clearly T).

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[**Base case**] $P(0) : f(0) = 0^2$ (clearly T).

[**Induction**] Show $P(n) \to P(n+1)$ for $n \ge 0$.

Using Induction to Analyze a Recursion

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Using Induction to Analyze a Recursion

$$f(n) = \begin{cases} 0 & n \le 0; \\ f(n-1) + 2n - 1 & n > 0. \end{cases}$$

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$f(n)$	0	1	4	9	16	25	36	49	64	$\cdots$

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Hard Example: A halving recursion (see text)

$$f(n) = \begin{cases} 1 & n = 1; \\ f(\frac{n}{2}) + 1 & n > 1, \text{ even}; \\ f(n+1) & n > 1, \text{ odd}; \end{cases}$$

(Looks esoteric? Often, you halve a problem (if it is even) or pad it by one to make it even, and then halve it.)

Prove $f(n) = 1 + \lceil \log_2 n \rceil$.

Practice. Exercise 7.5

Checklist for Analyzing Recursion

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Practice. Exercise 7.6

Recurrences: Fibonacci Numbers

Growth rate of rabbits, Sanskrit poetry, family trees of bees, ...

$$F_1 = 1;\ F_2 = 1; \qquad F_n = F_{n-1} + F_{n-2} \quad \text{for } n > 2.$$

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Practice. Prove $F_n \ge (\frac{3}{2})^n$ for $n \ge 11$.

Recursive Programs

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out=Big(n)
 if(n==0) out=1;
 else out=2*Big(n-1);
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Does this function compute 2^n?

Recursive Programs

Proving correctness: let's prove $\texttt{Big}(n) = 2^n$ for $n \geq 1$

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$$\begin{aligned} T_0 &= 2 \\ T_n &= T_{n-1} + (\text{check } \texttt{n==0}) + (\text{multiply by 2}) + (\text{assign to } \texttt{out}) \end{aligned}$$

Recursive Programs

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$$\begin{aligned} T_0 &= 2 \\ T_n &= T_{n-1} + (\text{check } \texttt{n==0}) + (\text{multiply by } 2) + (\text{assign to } \texttt{out}) \\ &= T_{n-1} + 3 \end{aligned}$$

Recursive Programs

```
out=Big(n)
 if(n==0) out=1;
 else out=2*Big(n-1);
```

Does this function compute 2^n?

Proving correctness: let's prove $\texttt{Big}(n) = 2^n$ for $n \geq 1$

Induction.

When $n = 0$, $\texttt{Big}(0) = 1 = 2^0$ ✓

Assume $\texttt{Big}(n) = 2^n$ for $n \geq 0$

$$\texttt{Big}(n+1) = 2 \times \texttt{Big}(n) = 2 \times 2^n = 2^{n+1}.$$

What is the runtime?

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Exercise. Prove by induction that $T_n = 3n + 2$.

Recursive Sets: $\mathbb{N}$

Recursive definition of the natural numbers $\mathbb{N}$.

1. $1 \in \mathbb{N}$. **[basis]**

$$\mathbb{N} = \{1,$$

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Technically, by bullet 3, we mean that $\mathbb{N}$ is the *smallest* set satisfying bullets 1 and 2.

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Technically, by bullet 3, we mean that $\mathbb{N}$ is the *smallest* set satisfying bullets 1 and 2.

Pop Quiz. Is $\mathbb{R}$ a set that satisfies bullets 1 and 2 alone? Is it the smallest?

Recursive Sets: Finite Binary Strings, Σ^*

Let ε be the *empty string* (similar to the empty set).

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ε

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$$\varepsilon \to 0, 1$$

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$$\varepsilon \rightarrow 0, 1 \rightarrow 00, 01, 10, 11$$

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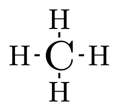
$$\Sigma^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, \ldots\}$$

Practice. Exercise 7.12

Recursive Structures: Trees

Sir Aurthur Cayley discovered trees when modeling chemical hydrocarbons,

methane, CH_4



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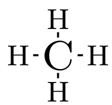
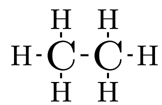


ethane, C_2H_6



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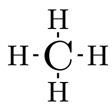

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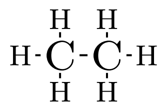
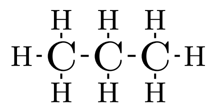


propane, C_3H_8



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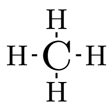

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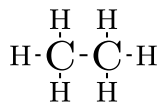

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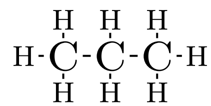
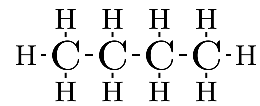


butane, C_4H_{10}



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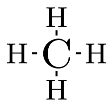

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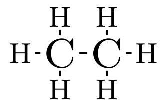

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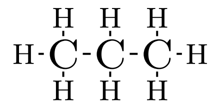
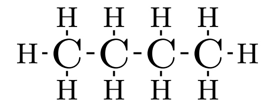
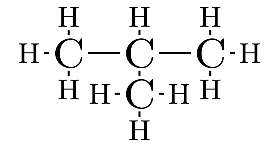


butane, C_4H_{10}



iso-butane, C_4H_{10}



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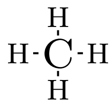

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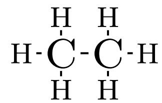

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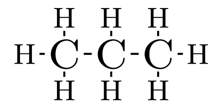

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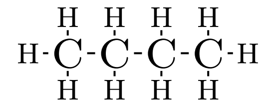

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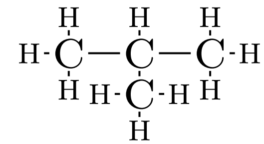

Trees have many uses in computer science

- Search trees.
- Game trees.
- Decision trees.
- Compression trees.
- Multi-processor trees.
- Parse trees.
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- ...

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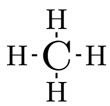

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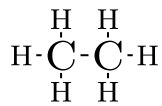

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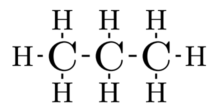

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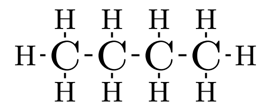
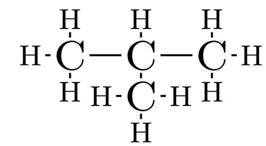


iso-butane, C_4H_{10}



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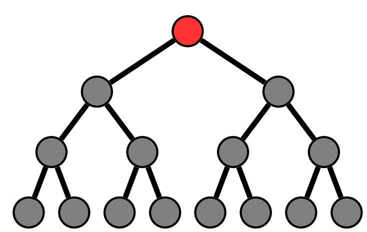
Tree.



Not a tree.

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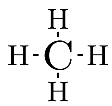

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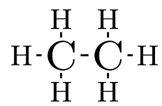

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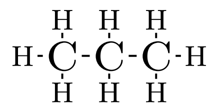

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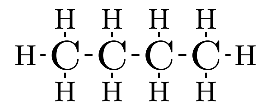

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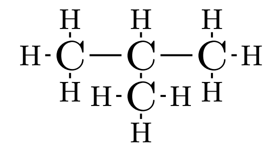

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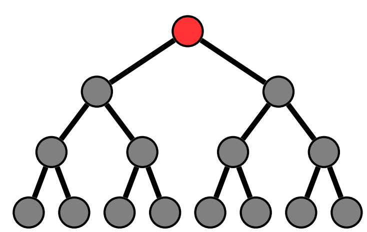

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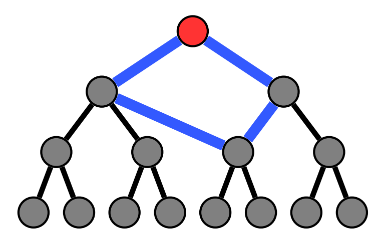

Rooted Binary Trees (RBT)

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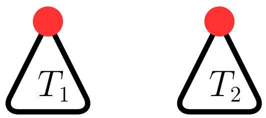

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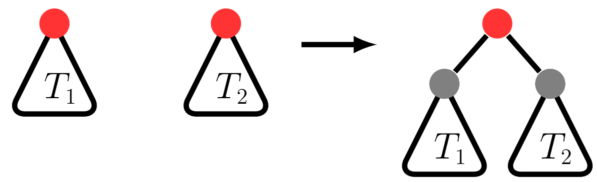

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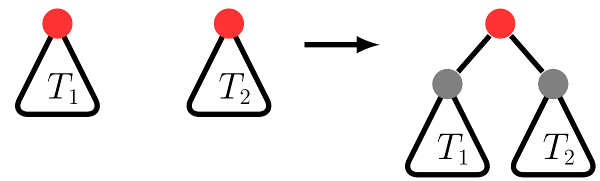

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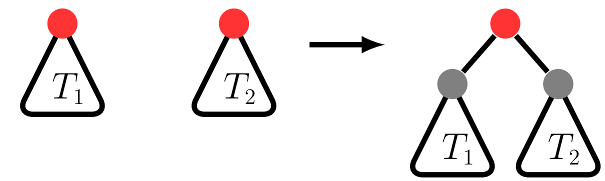

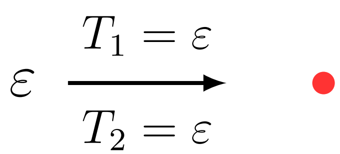

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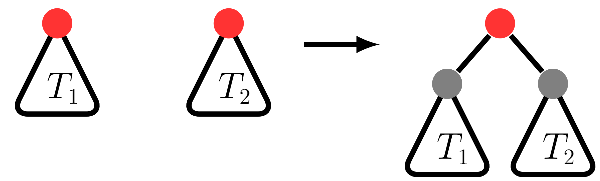

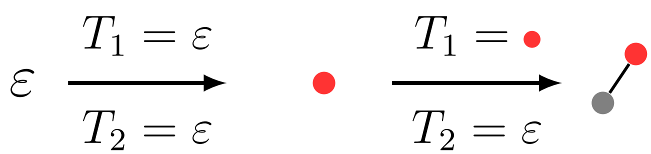

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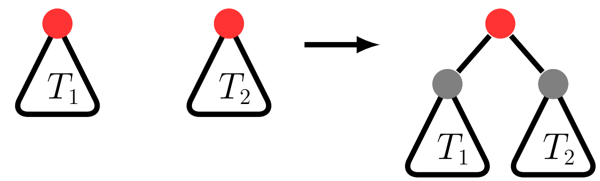

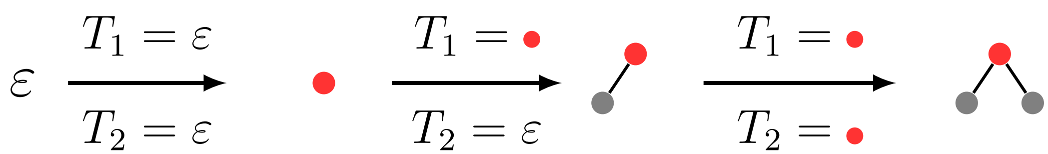

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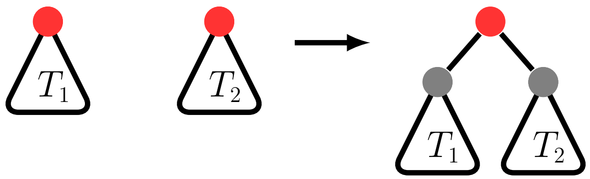

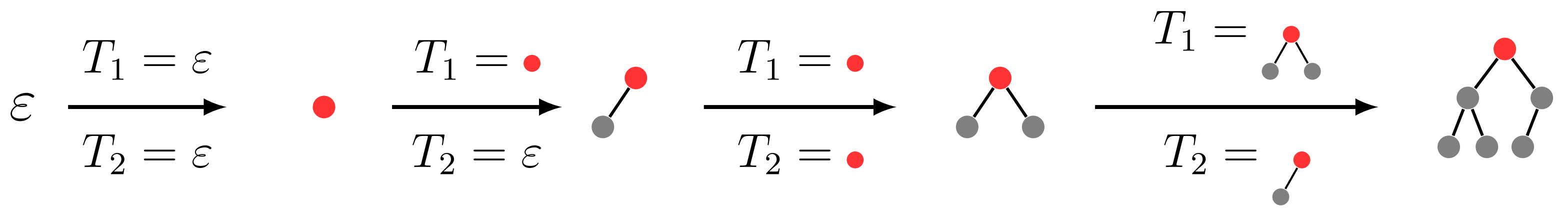

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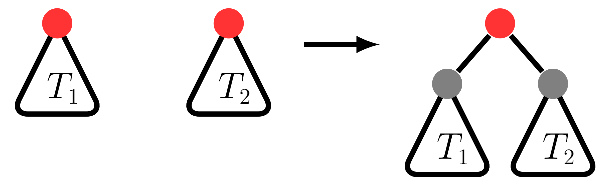

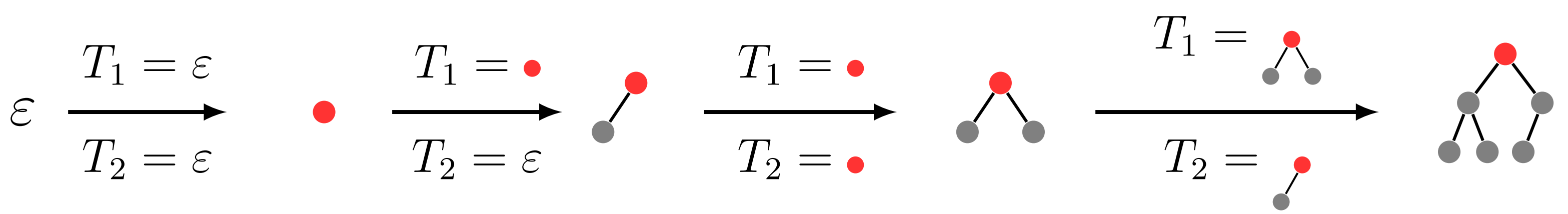

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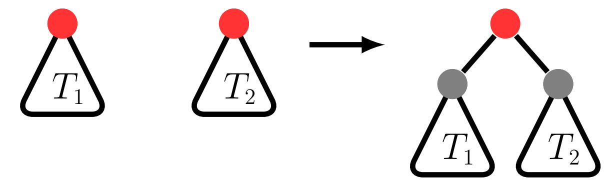

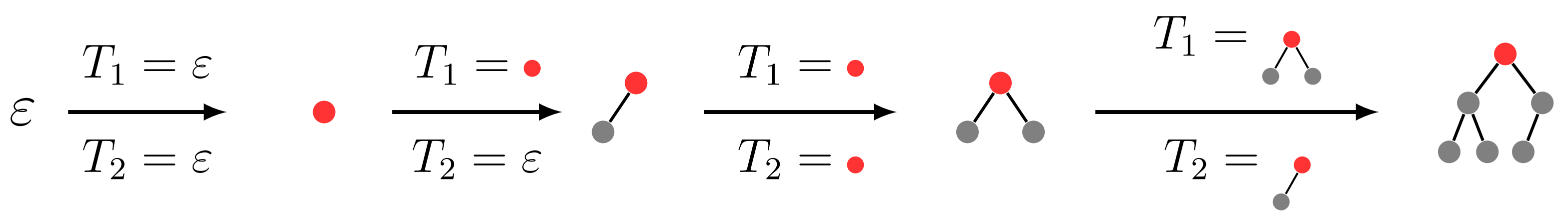

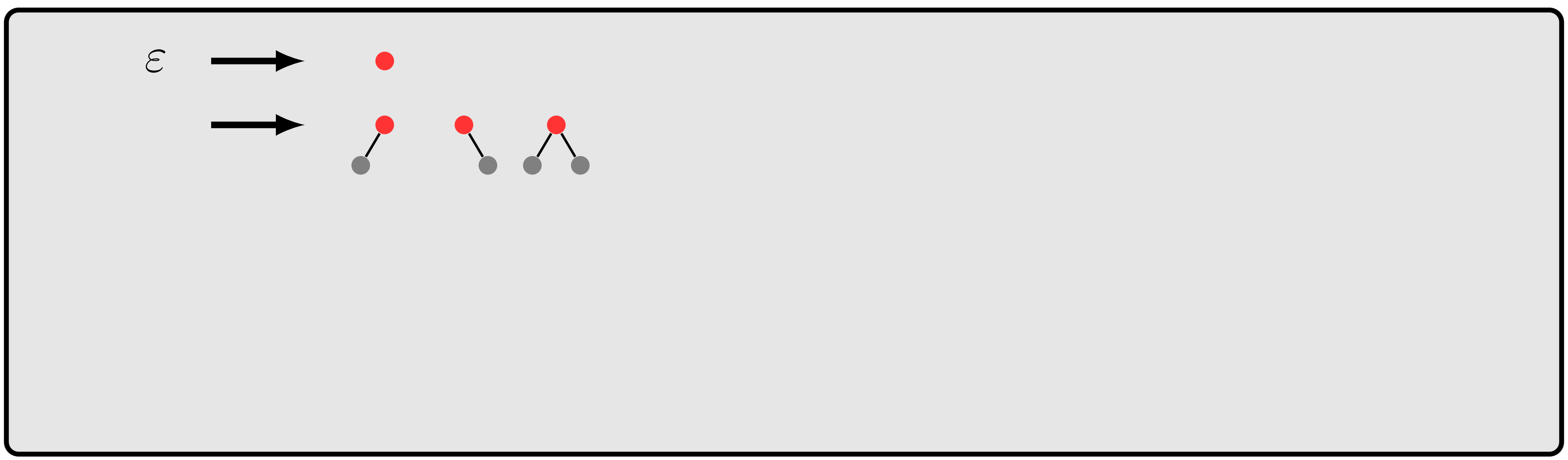

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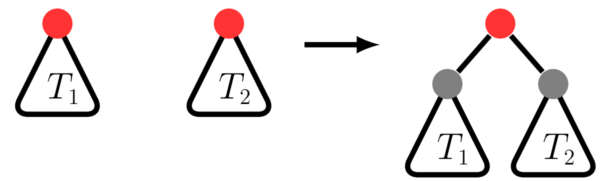

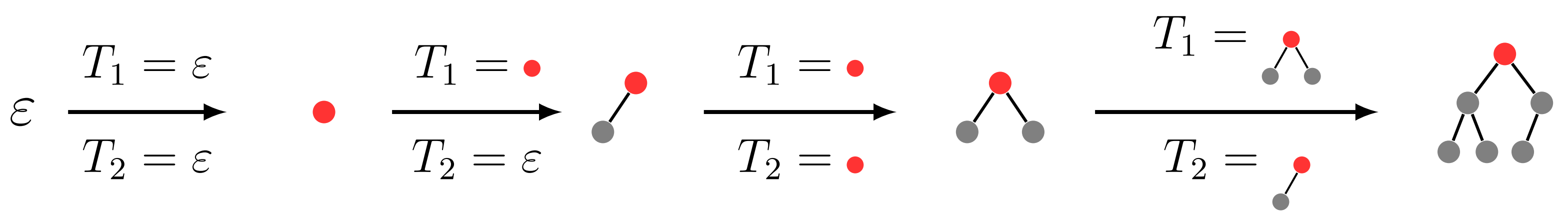

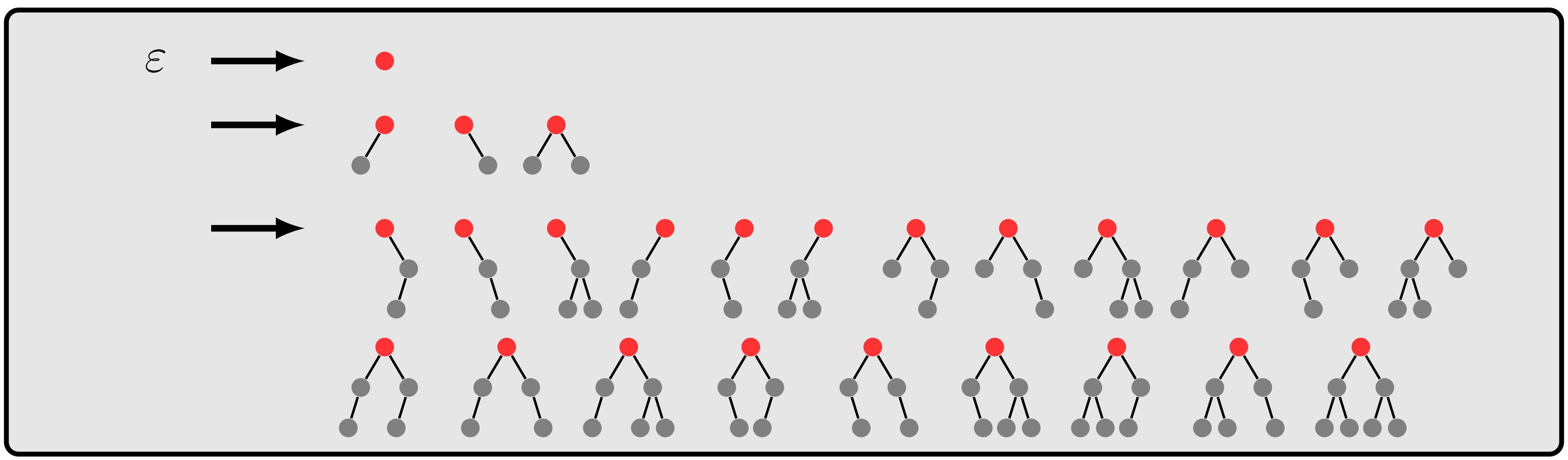

Trees Are Important: Food for Thought

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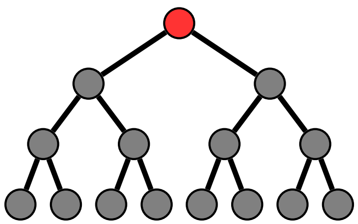
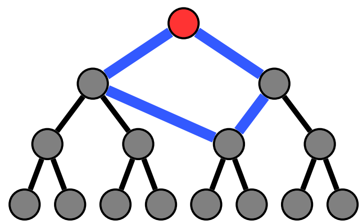


Not a tree.



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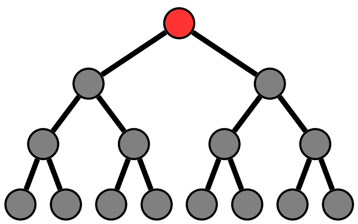

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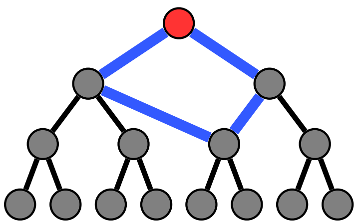

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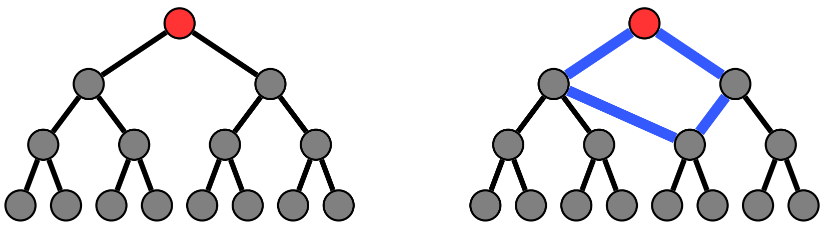

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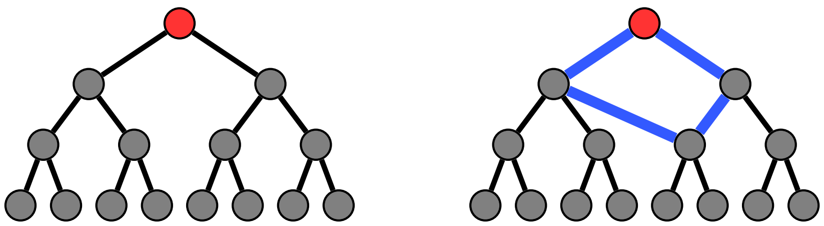

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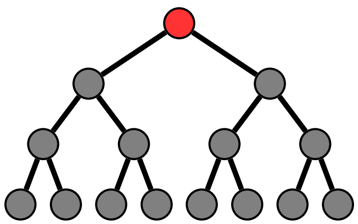

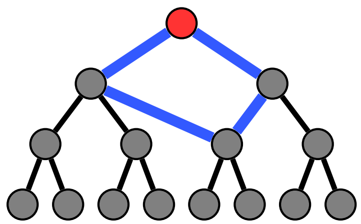

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  Can we be sure *every* RBT has these properties?