QUIZ 3: 110 Minutes

Solution

Answer **ALL** questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

You **MUST** show **CORRECT** work to get full credit.

When in doubt, TINKER.

1	2	3	Total
150	25	25	200

1 **Circle one answer per question.** 15 **points for each correct answer.**

(a) Let $\mathbf{X}$ and $\mathbf{Y}$ be independent random variables taking values in a set with n elements. What is the probability of the event $\mathbf{X} = \mathbf{Y}$?

- $\boxed{\text{A}}$ $1/n$
- $\boxed{\text{B}}$ $2/n$
- $\boxed{\text{C}}$ $1/n^2$
- $\boxed{\text{D}}$ $2/n^2$
- $\boxed{\text{E}}$ Not enough information/none of the above

Assume WLOG that set is $\{1, \dots, n\}$ and $\mathbb{P}(X=i)=p_i$, then by independence

$$\mathbb{P}(X=Y) = \sum_{i=1}^{n} \mathbb{P}(X=Y=i)$$
$$= \sum_{i=1}^{n} \mathbb{P}(X=i)\mathbb{P}(Y=i)$$
$$= \sum_{i=1}^{n} p_i^2$$

so depends on the distribution of X

(b) If $\mathbf{X}$ and $\mathbf{Y}$ are independent and have the same variance, which of these have the same variance as $\mathbf{X}$?

(I) $(\mathbf{X} - \mathbf{Y})/\sqrt{2}$
(II) $-\mathbf{X}$
(III) $(\mathbf{X} + \mathbf{Y})/2$

- $\boxed{\text{A}}$ I, II
- $\boxed{\text{B}}$ II, III
- $\boxed{\text{C}}$ I
- $\boxed{\text{D}}$ II
- $\boxed{\text{E}}$ III

$$\sigma^2\left(\frac{X-Y}{\sqrt{2}}\right) = \frac{1}{2}\sigma^2(X-Y)$$
$$= \frac{1}{2}\left[\sigma^2(X) + \sigma^2(Y)\right]$$
$$= \frac{1}{2}\left[2\sigma^2(X)\right] = \sigma^2(X)$$
$$\sigma^2(-X) = (-1)^2\sigma^2(X) = \sigma^2(X)$$
$$\sigma^2\left(\frac{X+Y}{2}\right) = \frac{1}{4}\left(\sigma^2(X+Y)\right) = \frac{1}{4}\left[\sigma^2(X) + \sigma^2(Y)\right]$$
$$= \frac{1}{4}\left[2\sigma^2(X)\right] = \frac{1}{2}\sigma^2(X)$$

(c) Which of the following is true? (All complements are taken in Σ^*.)

- $\boxed{\text{A}}$ The complement of a regular language may not be a regular language
- $\boxed{\text{B}}$ The complement of a language may not be a language
- $\boxed{\text{C}}$ The set of all infinite-length binary strings has smaller cardinality than the set of valid C programs
- $\boxed{\text{D}}$ The set of questions that can be asked using the English language is countable
- $\boxed{\text{E}}$ All of the above claims are true

F: regular languages are closed under complementation

F: $\bar{\mathcal{L}} \subseteq \Sigma^*$ so is a language

F: the other way around

T: there is an injection from these questions to Σ^* (use ASCII encoding, for example)

(d) Roll a six-sided die until you get a number greater than two. What is the expected number of rolls you need?

A 1.5

B 2

C 3.5

D 4

E None of the above.

$X = \#$ rolls to get a 3, 4, 5, or 6
has waiting time distrib w/ param $p = \frac{4}{6} = \frac{2}{3}$
so $\mathbb{E}X = \frac{1}{p} = \frac{3}{2}$

(e) Which of the following is not countable?

countable: use the same bijection construction as used to show ① is countable

A The Cartesian product $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$ of two countable sets

B The set of real solutions to the equation $\sin(x) = 0$ ← $x \in \{n\pi : n \in \mathbb{Z}\}$ ← countable

C The set of all graphs (use the definition of a graph in terms of vertices and edges)

D The set of all languages (use the definition of "language" from Chapter 23)

E The set of prime numbers

subset of $\mathbb{N}$, countable

power set of Σ^*, uncountable

can describe each graph w/ a finite length binary string, so countable

(f) Which of the following strings is not in the language given by the regular expression $(\{0\} \bullet \{11\}^*) \cup \{01\}^*$?

A ε ← is in $\{01\}^*$

B 001 ← is not in either $\{0\} \bullet \{11\}^*$ or $\{01\}^*$

C 0101 ← is in $\{01\}^*$

D 01111 ← is in $\{0\} \bullet \{11\}^*$

E All of the above are in the language

(g) You toss m balls randomly into n bins. Let $\mathbf{X}$ be the number of bins that contain exactly k balls. What is $\mathbb{E}[\mathbf{X}]$?

$\boxed{\text{A}}$ $n\frac{k!(m-k)!}{m!}$

$\boxed{\text{B}}$ n/k

$\boxed{\text{C}}$ $n\binom{m}{k}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{m-k}$

$\boxed{\text{D}}$ $n\frac{\binom{m}{k}}{\binom{m}{n}}$

$\boxed{\text{E}}$ None of the above.

$X = \sum_{i=1}^{n} X_i$ where $X_i = \begin{cases} 1 & \text{if bin } i \text{ has } k \text{ balls} \\ 0 & \text{otherwise} \end{cases}$

$\mathbb{E}X = \sum_{i=1}^{n} \mathbb{E}X_i = n\mathbb{E}X_1 = n\mathbb{P}(\text{bin 1 has } k \text{ balls})$

Consider each ball toss to be a trial that succeeds if ball lands in bin 1, with prob $\frac{1}{n} = p$

The prob of exactly k successes is $\binom{m}{k}p^k(1-p)^{m-k}$

so $\mathbb{E}X = n\binom{m}{k}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{m-k}$

(h) 100 student throw their graduation caps into the air, and the caps land randomly on the heads of the students. Let $\mathbf{X}$ be the number of students who get back their own hats. Given that both the variance and mean of $\mathbf{X}$ are 1, use Chebyshev's inequality to find an upper bound on the probability that $\mathbf{X}$ is at least 50.

$\boxed{\text{A}}$ 1/100

$\boxed{\text{B}}$ 1/50

$\boxed{\text{C}}$ 1/49

$\boxed{\text{D}}$ 1/32

$\boxed{\text{E}}$ None of the above

$\mathbb{P}(X \geq 50) = \mathbb{P}(X-1 \geq 49)$

$\leq \mathbb{P}(|X-1| \geq \sigma(X)\cdot 49)$

$\leq \frac{1}{49^2}$

(i) Count the number of injections from $\{1,2,3,4\}$ to $\{a,b,c,d,e\}$.

$\boxed{\text{A}}$ 9

$\boxed{\text{B}}$ 20

$\boxed{\text{C}}$ 4^5

$\boxed{\text{D}}$ $4!\binom{5}{4}$

$\boxed{\text{E}}$ $4!\binom{9}{2}$

Count the number of ways to assign 4 unique elements of $\{a,b,c,d,e\}$ to $\{1,2,3,4\}$:

$\binom{5}{4}\cdot 4!$

first choose 4 elements

then assign them to $\{1,2,3,4\}$ in this many different ways

(j) Kofi is considering vacationing in the Bahamas and is attempting to determine how many days of rain he should expect. His friend tells him that he can expect to wait 5 days between rainy days in the Bahamas. How many days of rain should Kofi expect during a 2 week stay in the Bahamas?

$\boxed{\text{A}}$ $1\frac{1}{5}$

$\boxed{\text{B}}$ $2\frac{4}{5}$

$\boxed{\text{C}}$ $3\frac{1}{2}$

$\boxed{\text{D}}$ 4

$\boxed{\text{E}}$ None of the above

Let p be the probability that it rains on a given day. The expected waiting time for rain is $\frac{1}{p} = 5$, so $p = \frac{1}{5}$. The number of days of rain in two weeks has distribution Binomial$(14, p)$, so has expectation $14p = \frac{14}{5} = 2\frac{4}{5}$

2 **For a fair coin, compute the expected number of flips to get 2 heads in a row.**

Let X be number of flips to get 2 heads in a row.

By total expectation,

$$\mathbb{E}[X] = \mathbb{E}[X \mid \text{roll a head}] \cdot \mathbb{P}(\text{roll a head}) + \mathbb{E}[X \mid \text{roll a tail}] \cdot \mathbb{P}(\text{roll a tail})$$

$$= \frac{1}{2}\mathbb{E}[X \mid \text{roll a head}] + \frac{1}{2}(1 + \mathbb{E}[X])$$

so

$$\mathbb{E}[X] = \mathbb{E}[X \mid \text{roll a head}] + 1$$

and

$$\mathbb{E}[X \mid \text{roll a head}] = 1 + 1 \cdot \mathbb{P}(\text{roll a head}) + (1 + \mathbb{E}[X]) \cdot \mathbb{P}(\text{roll a tail})$$

$$= 2 + \frac{1}{2}\mathbb{E}[X]$$

so

$$\mathbb{E}[X] = 3 + \frac{1}{2}\mathbb{E}[X],$$

and

$$\mathbb{E}[X] = 6$$

3 **Prove or disprove: the union of uncountably many languages is a language.**

Each language is a subset of Σ^*, so regardless of the number of them, their union is also a subset of Σ^*, so it is a language.

SCRATCH

SCRATCH