QUIZ 1: 60 Minutes

Solution

Answer **ALL** questions.

NO COLLABORATION or electronic devices. Any violations result in an F.

NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

You **MUST** show **CORRECT** work to get full credit.

When in doubt, TINKER.

1	2	3	Total
100	25	25	150

1 Circle one answer per question. 20 points for each correct answer.

(a) Identify the converse of the statement "If $\underbrace{\text{I have time and I am not too tired}}_{p}$, then $\underbrace{\text{I will go to the gym}}_{q}$".

- [A] If I will not go to the gym, then I have time and I am not too tired. p
- [B] If I will not go to the gym, then I do not have time or I am too tired.
- [C] If I will not go to the gym, then I do not have time and I am too tired.
- [D] If I will go to the gym, then I have time and I am not too tired.
- [E] None of the above.

Converse of $p \to q$ is $q \to p$

(b) Which type of proof is most appropriate to establish that the product of an irrational number and a rational number is irrational?

- [A] Direct.
- [B] Contraposition.
- [C] Induction.
- [D] Contradiction.
- [E] None of the above.

Let x be irrational and y rational. Assume $p = xy$ is rational, then $x = \frac{p}{y}$ is rational. This is a contradiction, so $p = xy$ is irrational

(c) Which of the following claims are true?

(1) $\neg(\forall x : P(x)) \overset{eqv}{\equiv} \exists x : \neg P(x)$

correct way of negating a qualifier

~~(2)~~ $P(x) \vee Q(x) \vee \neg P(x) \overset{eqv}{\equiv} Q(x)$

$P(x) \vee \neg P(x)$ is T so the left-hand side is $T \vee Q(x) = T$

~~(3)~~ $p \to q \overset{eqv}{\equiv} \neg q \vee p$

- [A] (1).
- [B] (1) and (2).
- [C] (1) and (3).
- [D] (2).
- [E] (2) and (3).

Recall $p \to q \overset{eqv}{\equiv} \neg p \vee q$ is the correct equivalence, alternatively, look at a truth table

p	q	$p \to q$	$\neg q$	$\neg q \vee p$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	T
T	T	T	F	T

we see $p \to q$ is not equiv to $\neg q \vee p$

(d) Which method of proof is most appropriate for establishing that for all positive integers $n \geq 2$, $1 + 2^n < 3^n$?

P(n)

- $\boxed{\text{A}}$ Contradiction.
- $\boxed{\text{B}}$ Contraposition.
- $\boxed{\text{C}}$ Direct.
- $\boxed{\text{D}}$ Induction.
- $\boxed{\text{E}}$ None of the above.

Prf

Base case when $n=2$, $1+2^n = 5 < 3^2 = 9$ so claim is true.

Induction Step Assume $1+2^n < 3^n$ for an $n \geq 2$, then $1+2^{n+1} = 1+2\cdot 2^n < 3\cdot 2^n < 3\cdot(1+2^n)$ because $1 < 2 < 2^n$. By the induction hypoth, $1+2^{n+1} < 3\cdot(1+2^n) < 3\cdot 3^n = 3^{n+1}$.

So $P(n) \to P(n+1)$. By induction, $P(n)$ is true when $n \geq 2$.

(e) What is the contrapositive of the statement "If $\underbrace{\text{I drive without first warming up the car}}_{p}$, $\underbrace{\text{the engine doesn't start or it shakes}}_{q}$"?

- $\boxed{\text{A}}$ If the engine doesn't start or it shakes, then I drove without first warming up the car.
- $\boxed{\text{B}}$ If the engine starts and doesn't shake, then I drove without first warming up the car.
- $\boxed{\text{C}}$ If the engine starts and doesn't shake, then I drove after first warming up the car.
- $\boxed{\text{D}}$ If the engine doesn't start or it shakes, then I drove after first warming up the car.
- $\boxed{\text{E}}$ None of the above.

The contrapositive of $p \to q$ is $\neg q \to \neg p$

2 **Let x, y, a be positive numbers with $x \leq y$. Prove that**

$$\frac{x+a}{y+a} \geq \frac{x}{y}.$$

Prf

We use contradiction. Assume that instead

$$\frac{x+a}{y+a} < \frac{x}{y}.$$ Then we see that

$$(x+a)y < x(y+a),$$ (because the denominators are both positive)

or

$$xy + ay < xy + xa,$$

so

$$ay < xa$$

and

$$y < x.$$ (because a is positive)

This contradicts the fact that $x \leq y$.

Therefore it is the case that

$$\frac{x+a}{y+a} \geq \frac{x}{y}$$

∎

3 **Prove that the sum of any five consecutive natural numbers is divisible by 5.**

Prf

We use a direct proof. Denote an arbitrary five consecutive natural numbers by $n, \ldots, n+4$. Their sum is

$$n + (n+1) + (n+2) + (n+3) + (n+4)$$

$$= 5n + 10 = 5(n+2),$$

and is therefore divisible by 5, as claimed.

$\blacksquare$

SCRATCH

SCRATCH