MIDTERM: 90 Minutes

Solution

Last Name: ____________________

First Name: ____________________

RIN: ____________________

Section: ____________________

Answer **ALL** questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.
SUBMIT your crib sheet, with your name written on it.

GOOD LUCK!

You **MUST** show **CORRECT** work to get full credit.

When in doubt, TINKER.

1	2	3	4	Total
120	25	25	25	195

1 **Circle one answer per question.** 15 **points for each correct answer.**

(a) Which method of proof is most appropriate for establishing that $(a+b)^n \geq a^n + b^n$ when a, b are non-negative reals and $n \in \mathbb{N}$?

- $\boxed{\text{A}}$ Direct
- $\boxed{\text{B}}$ Contraposition
- $\boxed{\text{C}}$ Contradiction
- $\boxed{\text{D}}$ Induction
- $\boxed{\text{E}}$ Structural Induction

Induction Step:

$$(a+b)^{n+1} = (a+b)(a+b)^n$$

by inductive hypothesis $\rightarrow$

$$\geq (a+b)(a^n+b^n)$$
$$= a^{n+1} + ba^n + ab^n + b^{n+1}$$
$$\geq a^{n+1} + b^{n+1}$$

(b) Which is true of the quantity $\frac{n^4+3n}{n^2+7}$?

- $\cancel{\boxed{\text{A}}}$ It is $o(n^2)$
- $\cancel{\boxed{\text{B}}}$ It is $\omega(n^2)$
- $\cancel{\boxed{\text{C}}}$ It is $o(n)$
- $\boxed{\text{D}}$ It is $\omega(n)$
- $\boxed{\text{E}}$ None of the above.

$$\frac{n}{\left(\frac{n^4+3n}{n^2+7}\right)} = \frac{n^3+7n}{n^4+3n} \rightarrow 0 \text{ as } n \rightarrow \infty \Rightarrow \in \omega(n)$$

$$\frac{n^2}{\left(\frac{n^4+3n}{n^2+7}\right)} = \frac{n^4+7n^2}{n^4+3n} \rightarrow 1 \text{ as } n \rightarrow \infty \Rightarrow \in \Theta(n^2)$$

(c) Which of these properties does the gcd have?

✓ (I) If ℓ is a positive integer linear combination of m and n, then $\gcd(m,n) | \ell$ ← since it divides m, n, divides any integer combination

✗ (II) If $d_1 | m$ and $d_2 | n$ and d_1 and d_2 are relatively prime, then $d_1 d_2 | \gcd(m,n)$ → false: $3|6$ and $7|21$ and 3 & 7 are prime but $\gcd(6,21)=3$ and $3 \cdot 7 \nmid 3$

✓ (III) If k is a natural number, $\gcd(m^k, n^k) = \gcd(m,n)^k$

write $\gcd(m^k, n^k)$ in terms of shared prime factors of m^k and n^k and observe these are the shared prime factors of m & n, raised to k.

- $\boxed{\text{A}}$ I and II.
- $\boxed{\text{B}}$ I and III.
- $\boxed{\text{C}}$ II and III.
- $\boxed{\text{D}}$ II only.
- $\boxed{\text{E}}$ III only.

(d) It is now 7pm. Where is the hour hand in 2023^3 hours?

- $\boxed{\text{A}}$ 3pm.
- $\boxed{\text{B}}$ 8pm.
- $\boxed{\text{C}}$ 12am.
- $\boxed{\text{D}}$ 2am.
- $\boxed{\text{E}}$ None of the above.

$$\begin{array}{r} 84 \\ 24 \overline{)2023} \\ -192 \\ \hline 103 \end{array} \quad \text{and} \quad \begin{array}{r} 84 \\ \times 24 \\ \hline 336 \\ 1680 \\ \hline 2016 \end{array} \Rightarrow 2023 \equiv 7 \bmod 24$$

$$\Rightarrow 2023^3 \bmod 24$$
$$= 49 \cdot 7 \bmod 24$$
$$= 1 \cdot 7 \bmod 24 = 7$$

7 hours after 7pm is 2am

$\frac{1}{n}\sum_{i=1}^{n} d_i < 2 \Rightarrow \sum_{i=1}^{n} d_i < 2n \Rightarrow$ handshaking theorem

$2|E| < 2n$

$\Rightarrow |E| < n$

$\Rightarrow |E| = n-1$ b/c graph connected

$\Rightarrow$ tree

(e) Which of these claims are true?

✓ (I) A connected graph on n vertices with average degree strictly less than 2 is a tree.

✓ (II) There does not exist a friend network with 7 friends, each of whom knows 3 friends.

✗ (III) $K_{4,5}$ is a regular graph.

left vertices have degree 5
right " " " 4
so not regular

$\boxed{\text{A}}$ All of them.

$\boxed{\text{B}}$ None of them.

$\boxed{\text{C}}$ I and II.

$\boxed{\text{D}}$ I and III.

$\boxed{\text{E}}$ II and III.

By handshaking theorem
$2|E| = 7 \cdot 3 = 21$ impossible
so no such graph exists

(f) What is the most precise asymptotic behavior of $S = \sum_{i=1}^{n} \exp(i)$?

$\boxed{\text{A}}$ $S \in O(\exp(n))$

$\boxed{\text{B}}$ $S \in O(\exp(n^2))$

$\boxed{\text{C}}$ $S \in \Theta(\exp(n))$

$\boxed{\text{D}}$ $S \in \Theta(\exp(n^2))$

$\boxed{\text{E}}$ None of the above.

$S = \exp(1) + \cdots + \exp(n) \geq \exp(n)$

and $S = \sum_{i=1}^{n-1} \exp(i) + \exp(n) \leq \int_1^n \exp(x)\,dx + \exp(n)$

$= 2\exp(n) - \exp(1)$

so

$\exp(n) \leq S \leq 2\exp(n) \Rightarrow S \in \Theta(\exp(n))$

(g) Compute $\sum_{i=1}^{2n}(1+3i)$

$\boxed{\text{A}}$ $2n + 6n^2$

$\boxed{\text{B}}$ $3n + 6n^2$

$\boxed{\text{C}}$ $5n + 6n^2$

$\boxed{\text{D}}$ $6n + 5n^2$

$\boxed{\text{E}}$ None of the above.

$\sum_{i=1}^{2n}(1+3i) = 2n + 3\sum_{i=1}^{2n} i = 2n + 3\frac{(2n)(2n+1)}{2}$

$= 2n + 3n(2n+1) = 6n^2 + 5n$

(h) Let G be a connected planar graph on 10 vertices with 15 edges. How many faces does G have?

$\boxed{\text{A}}$ 7

$\boxed{\text{B}}$ 9

$\boxed{\text{C}}$ 11

$\boxed{\text{D}}$ 12

$\boxed{\text{E}}$ None of the above.

$v - e + f = 2$

$\Rightarrow 10 - 15 + f = 2$

$\Rightarrow f = 7$

2 **Let $\mathcal{P}$ be a recursively defined set: $(1,0) \in \mathcal{P}$ and $(x,y) \in \mathcal{P} \to (x+1, y+2) \in \mathcal{P}$. Prove that every point $(x,y) \in \mathcal{P}$ satisfies $y = 2x - 2$.**

We use structural induction.

The basis element $(1,0)$ satisfies $0 = 2 \cdot 1 - 2$.

Assume $(x,y) \in P$ satisfies $y = 2x - 2$, then the element in P constructed as

$(\tilde{x} = x+1, \tilde{y} = y+2)$ satisfies

$$2\tilde{x} - 2 = \underbrace{2x = y+2}_{\text{follows from } y=2x-2} = \tilde{y}, \quad \text{or} \quad \tilde{y} = 2\tilde{x} - 2.$$

Thus by structural induction, every point $(x,y) \in P$ satisfies $y = 2x - 2$.

$\blacksquare$

3 **Show that when k is a natural number, $2^k - 1$ and $2^k + 1$ are relatively prime.**

We use a direct proof.
Let $p \in \mathbb{N}$ be a common divisor of $2^k - 1$ and $2^k + 1$, then because

$$(2^k + 1) - (2^k - 1) = 2,$$

p also divides 2. Thus $p = 2$ or $p = 1$.
The numbers $2^k + 1$ and $2^k - 1$ are odd, so $p = 1$. Thus the gcd of $2^k - 1$ and $2^k + 1$ is 1, so they are relatively prime.

4 **Prove that given a graph $G = (V, E)$ with n vertices, one can partition V into two sets so that every vertex in a set has at least half of its neighbors in the same set.**

We use proof by contradiction, as in problem 11.31b in HW5.

There exists a partition of V into two sets L and R that has a minimal number of edges that cross the partition (an edge crosses the partition if it is of the form (u, v) where $u \in L$ and $v \in R$).

Assume that this partition does not have the property that each vertex in L and R has at least half of its neighbors in the same set. Then there exists a vertex v s.t. it has more neighbors in the other set than it does in its own set. Without loss of generality, $v \in L$. Let d_L be the number of neighbors of v in L and d_R be the number of neighbors of v in R. Notice that v has d_R edges crossing the partition.

CONT'D →

SCRATCH

Consider the partition obtained by swapping v over to R. Then v now has d_L edges crossing the partition. By assumption, $d_R > d_L$, so this new partition has fewer edges crossing the partition. This contradicts the fact that the partition of V into L and R minimizes the number of edges crossing the partition.

Thus we conclude that the partition of V into L and R has the desired property that each vertex has at least half of its neighbors in the same set.

$\square$

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