# **FINAL:** 180 Minutes

Last Name: Solution

First Name: ______________________

RIN: ______________________

Section: ______________________

Answer **ALL** questions.
**NO COLLABORATION or electronic devices. Any violations result in an F.**
**NO questions** allowed during the test. Interpret and do the best you can.

# GOOD LUCK!

You **MUST** show **CORRECT** work to get full credit.

**When in doubt, TINKER.**

| 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|
| | | | | | | |
| 200 | 30 | 30 | 30 | 30 | 30 | 350 |

# 1 Circle one answer per question. 10 points for each correct answer.

(1) If $P(3)$ is true and $P(n) \to (P(n^2) \wedge P(n-2))$ for $n \geq 2$, which of the following can I conclude?

- A $P(120)$ is true
- B $P(46)$ is not true
- C $P(1)$ is not true
- D $P(111)$ is true
- E None of the above

$P(3) \to P(9) \wedge P(1)$

$P(9) \to P(81) \wedge P(7)$

$P(7) \to P(49) \wedge P(5)$

Notice $P(5) \to P(25) \to P(23) \to \cdots \to P(11) \to P(121)$
$\to P(119) \to \cdots \to P(111)$

1 2 3 4 5 6 7 8 9

(2) What is the most precise asymptotic relationship between $\sum_{i=1}^{n} i\sqrt{i}$ and $n^2$?

- A $\sum_{i=1}^{n} i\sqrt{i} \in O(n^2)$
- B $\sum_{i=1}^{n} i\sqrt{i} \in \Omega(n^2)$
- C $\sum_{i=1}^{n} i\sqrt{i} \in \Theta(n^2)$
- D $\sum_{i=1}^{n} i\sqrt{i} \in o(n^2)$
- E $\sum_{i=1}^{n} i\sqrt{i} \in \omega(n^2)$

$f(x) = x^{3/2}$ is increasing in $x$, so

$$\int_0^n x^{3/2}\,dx \leq \sum_{i=1}^{n} i^{3/2} \leq \int_1^{n+1} x^{3/2}\,dx = \frac{2}{5}(n+1)^{5/2} - \frac{2}{5}$$

$$\int_0^n x^{3/2}\,dx = \frac{2}{5}n^{5/2}$$

$$\Rightarrow \sum_{i=1}^{n} i\sqrt{i} \in \Theta(n^{5/2}) \Rightarrow \sum_{i=1}^{n} i\sqrt{i} \in \omega(n^2)$$

(3) In an ESP experiment, a pair of fair six-sided dice is rolled in a separate room and the supposed esper states that at least one of the dice came up six. What is the probability that the sum of the two dice is seven?

- A 2/11
- B 1/9
- C 2/7
- D 3/5
- E None of the above

$$\mathbb{P}[X+Y=7 \mid X=6 \text{ or } Y=6] = \frac{\mathbb{P}[\{X+Y=7\} \cap \{X=6 \text{ or } Y=6\}]}{\mathbb{P}[X=6 \text{ or } Y=6]}$$

$$= \frac{\mathbb{P}[X=6, Y=1] + \mathbb{P}[X=1, Y=6]}{\mathbb{P}[X=6] + \mathbb{P}[Y=6] - \mathbb{P}[X=6, Y=6]}$$

$$= \frac{1/36 + 1/36}{1/6 + 1/6 - 1/36} = \frac{2}{11}$$

(4) What is the coefficient of $x^{10}$ in $(x - x^{-1})^{20}$?

- A $-\binom{20}{15}$
- B $\binom{20}{5}$
- C $-\binom{20}{10}$
- D $\binom{20}{10}$
- E None of the above

$$(x - x^{-1})^{20} = \sum_{j=0}^{20} \binom{20}{j} \left(-\frac{1}{x}\right)^j (x)^{20-j}$$

$$= \sum_{j=0}^{20} (-1)^j \binom{20}{j} x^{20-2j}$$

when $j = 5$, $(-1)\binom{20}{5} = (-1)\binom{20}{15}$

(5) What is the last digit of $3^{201}$?

- A 1
- B 3
- C 7
- D 9
- E None of the above

$$3^{201} \equiv (3^{200} \bmod 100) \cdot 3 \bmod 100$$

$$\equiv (9 \bmod 100)^{100} \cdot 3 \bmod 100$$

$$\equiv 3 \bmod 100$$

(6) Which of the following logical equivalences are correct?

(I) $p \vee q \to r \overset{\text{eqv}}{\equiv} \neg r \to \neg p \vee \neg q$ ✗

(II) $p \vee (q \wedge r) \overset{\text{eqv}}{\equiv} (p \wedge q) \vee (p \wedge r)$ ✗

(III) $(\neg q \vee r) \vee \neg p \overset{\text{eqv}}{\equiv} \neg q \vee (r \vee \neg p)$ ✓ associativity of $\vee$

A I

B II

C III

D I, II

E I, III

(7) Let the number of courses offered in the CSCI department be 27, and the number of courses offered in the math department be 25. If there are 5 courses cross-listed in both departments, how many different ways can we choose three distinct courses if those courses can be from either department?

There are $27+25-5=47$ unique courses by PIE

A $\binom{25}{3}$

B $\binom{27}{3}$

C $\binom{47}{3}$

D $\binom{52}{3}$

E None of the above

(8) On a multiple choice test with 20 questions and 5 answers per question, how many different ways can the test be completed so that exactly half of the answers are wrong?

A $\binom{20}{10}\left(\frac{1}{5}\right)^{10}\left(\frac{4}{5}\right)^{10}$

B $\binom{20}{10}\left(\frac{4}{5}\right)^{10}$

C $\binom{20}{10}$

D $\binom{20}{10}10^4$

E $\binom{20}{10}4^{10}$

$\binom{20}{10}$ ways to choose which half of the problems to get incorrect, and for each of the 10 incorrect answers, 4 choices that make them incorrect

(9) Which set is *not* countable?

A The set of polynomials with rational coefficients and degree at most $d$.

B The set of circles in the plane with centers $c = (x, y)$ and radii $r$ satisfying $x, y \in \mathbb{Z}$ and $r \in \mathbb{Q}$.

C The Cartesian product of two countable sets.

D The set of finite subsets of $\mathbb{Q}$.

E The set of irrational numbers.

(10) The graph $G$ has six vertices with degrees $2, 2, 3, 4, 4, 5$. How many edges does $G$ have?

A 8

B 10

C 12

D 20

E None of the above

hand-shaking theorem

$\sum d_i = 2|E| \Rightarrow |E| = \frac{1}{2}(2+2+3+4+4+5)$

$= 10$

(11) Which of these claims are true?

(I) If $x$ and $y$ are coprime, then $\mathbb{N} = \{mx + ny \mid m, n \in \mathbb{N}\}$.

(II) If $x$ and $y$ are coprime, then there is a $k \in \mathbb{N}$ for which $\gcd(xk, yk) \neq k$.

I is false (by Bezout $\mathbb{Z} = \{mx+ny \mid m,n \in \mathbb{N}$ if $\gcd(x,y) = 1$)

II is false (it is a gcd property that $\forall k \in \mathbb{N}: \gcd(xk, yk) = k\gcd(x,y)$)

A I only.

B II only.

C Both I and II.

D Neither I or II.

E The truth values of these claims depend on the specific values of $x$ and $y$.

(12) Which computing problem is context-free but not regular?

A $\mathcal{L} = \{\omega\#\omega \mid \omega \in \{0,1\}^*\}$ ← not context-free (cannot solve w/ PDA)

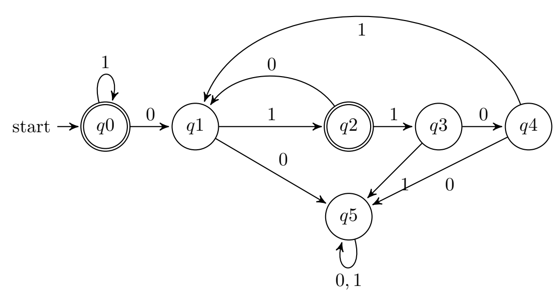
B $\mathcal{L} = \{\omega\#\omega^R \mid \omega \in \{0,1\}^*\}$ ← our first example of context-free and not regular

C $\mathcal{L} = \{1^{\bullet n}0^{\bullet n}1^{\bullet n} \mid n \geq 0\}$ ← not context-free

D $\mathcal{L} = \{1^{\bullet(2n)}01^{\bullet(2k+1)} \mid n, k \geq 0\}$ ← regular

E All of the above are regular languages

(13) Which string is accepted by this DFA?




A 1011

B 101000

C 101101

D 110011

E 1110101

(14) A team is three times more likely to win than lose each game it plays in a tournament. What is the expected number of games the team will play to get two wins?

A 8/3

B 3/2

C 2

D 3

E None of the above

$p = 3(1-p) \Rightarrow p = \frac{3}{4}$

Expectation of waiting time w/ param $p$ is $\frac{1}{p}$

Expectation of waiting until two wins is

$\frac{2}{p} = \frac{8}{3}$

(15) How many permutations of $\{1, \ldots, 7\}$ keep exactly four elements fixed (i.e. those elements are mapped to themselves under the permutation)? *Hint: equivalently, count the number of permutations that ensure that exactly three elements are not fixed.*

A 42

B 61

C 65

D 70

E None of the above

There are $\binom{7}{3}$ ways to choose three elements to derange, then 2 ways to derange 3 elements:

$1,2,3 \to 3,1,2$ or $2,3,1$

so $\binom{7}{3}2 = \frac{7!\,2}{3!\,4!} = \frac{7\cdot 6\cdot 5}{6}\cdot 2 = 70$

(16) A kindergarden teacher takes his class of ten children to the county fair and buys them cotton candy. The cotton candy machine has fifteen flavors. Each purchase yields a random flavor; each flavor has a non-zero probability of being selected. How many candies must the teacher buy to ensure all the children get the same flavor?

A 125

B 136

C 143

D 151

E None of the above

It's possible the first $15\cdot 9$ candies he buys consists of 9 candies of each colors. Once he buys $15\cdot 9+1$ candies, at least one color of candy will occur 10 times

(17) Giselle has fifty wigs, and a display cabinet with six shelves. How many ways can Giselle arrange her wigs on these shelves if the order of the wigs on each shelf does not matter, and each shelf can hold all fifty wigs?

A $50^6$

B $6^{50}$

C $\binom{50}{6}$

D $50!/6!$

E None of the above

For each of the 50 wigs she has six choices of shelves

(18) Evaluate the sum $T(n) = \sum_{i=1}^{n} 2^{3-2i}$ $= 8 \sum_{i=1}^{n} 2^{-2i} = 8 \sum_{i=1}^{n} \left(\frac{1}{4}\right)^i$

$= 8\left(\sum_{i=0}^{n}\left(\frac{1}{4}\right)^i - 1\right) = 8\left(\frac{1-\left(\frac{1}{4}\right)^{n+1}}{1-\frac{1}{4}} - 1\right)$

$= 8\left(\frac{4}{3}\left(1-\left(\frac{1}{4}\right)^{n+1}\right)-1\right) = 8\left(\frac{4}{3} - \frac{4}{3}\left(\frac{1}{4}\right)^{n+1} - 1\right)$

$= \frac{8}{3}\left(1-\left(\frac{1}{4}\right)^n\right)$

- [A] $\frac{2}{3}\left(4 - 4^{-n}\right)$
- [B] $\frac{8}{3}\left(1 - 4^{-n}\right)$
- [C] $\frac{8}{3}\left(4 - 4^{-n}\right)$
- [D] $\frac{32}{3}\left(1 - 4^{-n}\right)$
- [E] $\frac{32}{3}\left(4 - 4^{-n}\right)$

(19) Which of the following is true?

(I) $\{0, 1\}^*$ is decidable.
(II) If $\mathcal{L}$ is decidable and $\mathcal{S} \subseteq \mathcal{L}$, then $\mathcal{S}$ is decidable.
(III) If $\mathcal{L}$ is recognizable and $\mathcal{S} \subseteq \mathcal{L}$, then $\mathcal{S}$ is recognizable.

both false because we know there exist languages that are undecidable or unrecognizable, yet these are subsets of $\{0,1\}^*$, which is decidable

- [A] I only
- [B] II only
- [C] III only
- [D] I and III only
- [E] I, II, and III

(20) Which of these degree sequences is graphical? (Recall that a degree sequence is graphical iff there exists a graph with that degree sequence)

- [A] [3, 2, 2, 2]
- [B] [3, 3, 3, 1]
- [C] [3, 3, 3, 2]
- [D] [3, 3, 3, 3]  $K_4$
- [E] None of the above

**2** **Let $\mathcal{V}$ be a set of $n$ vertices, and let the edge set $\mathcal{E}$ be initially empty. For each pair of vertices $i \neq j$, add the edge $(i, j)$ to $\mathcal{E}$ with probability $p$. Let $\mathbf{X} = |\mathcal{E}|$ be the number of edges. Compute the expectation and variance of $\mathbf{X}$.**

$X = \sum_{\substack{(i,j) \in V^2 \\ i \neq j}} X_{ij}$, where the $X_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$

are $\binom{n}{2}$ indep Bern$(p)$ r.v.s

$\Rightarrow X \sim \text{Binomial}\left(\binom{n}{2}, p\right)$

$\Rightarrow \mathbb{E}X = \binom{n}{2} p$

$\sigma^2(X) = \binom{n}{2} p(1-p)$

**3** **Show that the recognizable languages are closed under intersection: let $\mathcal{L}_1$ and $\mathcal{L}_2$ be recognizable, and show that $\mathcal{L}_\cap = \mathcal{L}_1 \cap \mathcal{L}_2$ is recognizable by using the recognizers for $\mathcal{L}_1$ and $\mathcal{L}_2$ to construct a recognizer for $\mathcal{L}_\cap$. Prove that it is indeed a recognizer.**

Let $R_1$ be a recognizer for $\mathcal{L}_1$ and $R_2$ be a recognizer for $\mathcal{L}_2$.

Define a TM $R_\cap$ that computes

$$R_\cap(\omega) = \begin{cases} \text{ACCEPT} & \text{if } R_1(\omega) = R_2(\omega) = \text{ACCEPT} \\ \text{REJECT} & \text{if } R_1(\omega) = \text{REJECT or} \\ & \quad R_2(\omega) = \text{REJECT} \end{cases}$$

This TM simulates $R_1$ and $R_2$ on its input, so halts on inputs for which $R_1$ and $R_2$ both halt.

In particular, if $\omega \in \mathcal{L}_\cap$, then $\omega \in \mathcal{L}_1$ and $\omega \in \mathcal{L}_2$ so $R_1(\omega)$ and $R_2(\omega)$ both halt and ACCEPT. This means that if $\omega \in \mathcal{L}_\cap$, $R_\cap(\omega)$ will halt and ACCEPT, so $R_\cap$ is a recognizer for $\mathcal{L}_\cap$.

**4** **Consider the language of even-length palindromes $\mathcal{L} = \{\omega\omega^R \,|\, \omega \in \{0,1\}^*\}$. Give well-written pseudocode for a decider for this language.**

1. Check that the input has even length (including 0), otherwise REJECT. A DFA can do this.
2. Return to *
3. Move right to first non-marked bit. If you reach ⊔ before any non-marked bit, ACCEPT.
   Otherwise, mark the location and remember the bit.
4. Move right to ⊔ and then move left to the first non-marked bit. If the bit does not match the bit from step 3, REJECT.
   Otherwise (bit matches), mark the location.
   GOTO step 2.

**5** Consider the following process of assigning final projects to $k$ students. Each student is assigned their final project uniformly at random from $n$ possible projects ($n \geq k$), independent of the other students' assignments. If each student is assigned a unique project, then the project selection process finishes. If any two students are assigned the same project, the project selection process restarts. What is the expected number of times this project selection process will be used before the students all have unique projects?

To rephrase: what is the expected waiting time when a trial is successful iff each student is assigned a unique project?

There are $n^k$ ways to assign projects to all $k$ students, and $\binom{n}{k}k!$ ways to assign unique projects to the $k$ students, so the probability of success of a single trial is given by

$$p = \frac{\binom{n}{k}k!}{n^k}$$

(NB: I used the fact that we are in a uniform prob space in this counting argument)

It follows that the expected waiting time is

$$\boxed{\frac{n^k}{\binom{n}{k}k!}}$$

**6** **Solve the recurrence relation $T(n) = 2T(n-1) - T(n-2) + 2$ when $T(0) = 1$ and $T(1) = 2$.**

Notice that subtracting $T(n-1)$ from both sides of the recursion gives the identity

$$T(n) - T(n-1) = T(n-1) - T(n-2) + 2.$$

Letting $D(n) \equiv T(n) - T(n-1)$, we have that

$$D(1) = 1 \quad \text{and} \quad D(n) - D(n-1) = 2$$

from which we have

$$D(n) = [D(n) - D(n-1)] + [D(n-1) - D(n-2)] + \cdots + [D(2) - D(1)] + D(1)$$

$$= 2 \cdot (n-1) + 1$$

Likewise,

$$T(n) = [T(n) - T(n-1)] + [T(n-1) - T(n-2)] + \cdots + [T(1) - T(0)] + T(0)$$

$$= \sum_{i=1}^{n} D(i) + 1 = \sum_{i=1}^{n} [2 \cdot (i-1) + 1] + 1$$

$$= 2 \sum_{i=1}^{n-1} i + n + 1 = (n-1)n + (n+1)$$

$$= n^2 - n + n + 1 = n^2 + 1 \qquad \boxed{T(n) = n^2 + 1}$$

## SCRATCH

**SCRATCH**

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