FINAL: 180 Minutes

Last Name: Solution

First Name: ________________

RIN: ________________

Section: ________________

Answer **ALL** questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

You **MUST** show **CORRECT** work to get full credit.

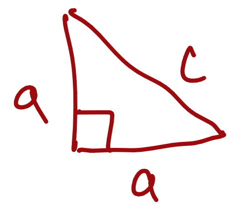
When in doubt, TINKER.

1-12	Total
10 each	120

1 **Determine the type of proof and prove: a right triangle with integer sides cannot be isoceles.**

Contradiction.

Let there be an isoceles right triangle with integer sides of lengths a and c.



Then $a^2+a^2=c^2$, so $2=\left(\frac{c}{a}\right)^2$, or $\sqrt{2}=\frac{c}{a}$.

This is a contradiction, as $\sqrt{2}$ is irrational. Therefore no such triangle exists.

2 **Prove that the product of any 5 consecutive natural numbers is divisible by 5!.**

Let $n, n+1, \dots, n+4$ be 5 consecutive natural numbers.

By the pidgeon hole principle, every 5 consecutive numbers contain a multiple of 5, because the multiples of 5 are within distance 5 of each other. This means one of $n, n+1, \dots, n+4$ is divisible by 5.

Similarly, multiples of 3 are within distance 3 of each other, so one of $n, n+1, \dots, n+4$ is divisible by 3.

Similarly multiples of 4 are within distance 4 of each other, so one of $n, n+1, \dots, n+4$ is divisible by 4. There is at least one <u>other</u> even number in the list $n, n+1, \dots, n+4$, as there are at least two even numbers in every list of five numbers.

Altogether, we can conclude that the product of $n, n+1, \dots, n+4$ is divisible by $5 \cdot 4 \cdot 3 \cdot 2 = 5!$

$\blacksquare$

Alternatively:

Write the consecutive numbers as

$$n+1, \dots, n+5.$$

Then

$$(n+1)\cdots(n+5) = \frac{(n+5)!}{n!} = 5!\left[\frac{(n+5)!}{n!\,5!}\right]$$

$$= 5!\binom{n+5}{n}$$

so we see directly that the product is divisible by $5!$

3 **Let $G_0 = 0$, $G_1 = 1$, and $G_n = 7G_{n-1} - 12G_{n-2}$ for $n > 1$. Compute G_5. Show $G_n = 4^n - 3^n$ for $n \geq 0$.**

$G_2 = 7G_1 - 12G_0 = 7$

$G_3 = 7G_2 - 12G_1 = 49 - 12 = 37$

$G_4 = 7G_3 - 12G_2 = 7 \cdot 37 - 12 \cdot 7 = 7 \cdot 25 = 175$

$G_5 = 7G_4 - 12G_3 = 7 \cdot 175 - 12 \cdot 37 = 1225 - 444 = 781$

$\boxed{G_5 = 781}$

Prf $G_n = 4^n - 3^n$ for $n \geq 0$.

We use strong induction

When $n=0$, $G_0 = 0 = 4^0 - 3^0$, so the formula holds.

Now assume that $G_k = 4^k - 3^k$ when $k \geq 0$ and $k \leq n$.

Then

$$G_{n+1} = 7G_n - 12G_{n-1} = 7 \cdot (4^n - 3^n) - 12(4^{n-1} - 3^{n-1})$$

by the inductive hypothesis, so

$$G_{n+1} = 7 \cdot (4^n - 3^n) - (3 \cdot 4^n - 4 \cdot 3^n)$$

$$= 4 \cdot 4^n - 3 \cdot 3^n$$

$$= 4^{n+1} - 3^{n+1}$$

so the formula holds for $n+1$.

We conclude that

$$G_n = 4^n - 3^n \quad \text{for } n \geq 0$$

by the principle of induction

$\boxtimes$

4 **Determine and prove the order-relationships between** $\ln n$, $\ln(n^2+1)$, **and** $\ln(2n)$.

- $\ln(2n) = \ln(2) + \ln(n) \in \Theta(\ln n)$
- $\ln(n^2+1) \in \Theta(\ln(n^2))$ and $\ln(n^2) = 2\ln(n)$,

so $\ln(n^2+1) \in \Theta(\ln n)$

Thus all three functions are asymptotically equivalent (Θ)

5 **Prove that**

$$xk \equiv yk \mod d$$

implies that

$$x \equiv y \mod (d/\gcd(k,d)).$$

Prf

Assume $xk - yk \equiv \bmod d$; this means $d \mid (x-y)k$. Now observe that both d and $(x-y)k$ are divisible by $\gcd(d,k)$, so

$$\frac{d}{\gcd(d,k)} \;\Big|\; (x-y)\frac{k}{\gcd(d,k)}.$$

Also note that by the definition of the gcd, the numbers $\frac{d}{\gcd(d,k)}$ and $\frac{k}{\gcd(d,k)}$ are coprime.

Since $\frac{d}{\gcd(d,k)} \Big| (x-y)\frac{k}{\gcd(d,k)}$ and does not divide $\frac{k}{\gcd(d,k)}$, we conclude that

$$\frac{d}{\gcd(d,k)} \;\Big|\; (x-y),$$

so

$$x \equiv y \mod \left(\frac{d}{\gcd(d,k)}\right)$$

$\blacksquare$

6 **Adam, Barb, Charlie, and Doris each independently choose a random number uniformly distributed in $\{1, 2, 3, 4, 5\}$. What is the probability that some pair chooses the same number? What if there are k people and n numbers?**

$\mathbb{P}$[some pair chooses the same number] =

1 − $\mathbb{P}$[everyone chooses a unique number]

$$= 1 - \frac{\binom{5}{4}4!}{5^4}$$

$\binom{5}{4}4!$ ← number of ways to assign four unique numbers of 5 possible numbers, to four people

5^4 ← number of ways to assign each person a number in 1 through 5.

In the general case,

$\mathbb{P}$[some pair chooses the same number] =

$$1 - \frac{\binom{n}{k}k!}{n^k}$$

7 **Let $\mathcal{V}$ be a set of n vertices, and let the edge set $\mathcal{E}$ be initially empty. For each pair of vertices $i \neq j$, add the edge (i, j) to $\mathcal{E}$ with probability p. Give the pdf for the degrees of the nodes of this graph.**

Each node is connected to each other node w/ prob p. Let X be the degree of a node, then

$$X \sim \text{Binomial}(n-1, p)$$

so the pdf is given by

$$\mathbb{P}[X=k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

for $k \in \{0, \dots, n-1\}$

8 **Voltage in the US has a mean of 120V and a standard deviation of 5V. A device's operating voltage is 112–128. Use Chebyshev's inequality to bound the probability that the device will not be damaged when turned on.**

The probability the device will not be damaged is

$$\mathbb{P}[V \in [112, 128]] = 1 - \mathbb{P}(V \notin [112, 128])$$

$$= 1 - \mathbb{P}(|V - 120| \geq 8)$$

where V is the voltage. Now applying Chebyshev:

$$\mathbb{P}(|V - 120| \geq 8) = \mathbb{P}(|V - \mathbb{E}V|^2 \geq 64)$$

$$= \mathbb{P}\left(|V - \mathbb{E}V|^2 \geq \frac{64}{25} \cdot 25\right)$$

$$= \mathbb{P}\left(|V - \mathbb{E}V|^2 \geq \frac{64}{25} \sigma^2(V)\right)$$

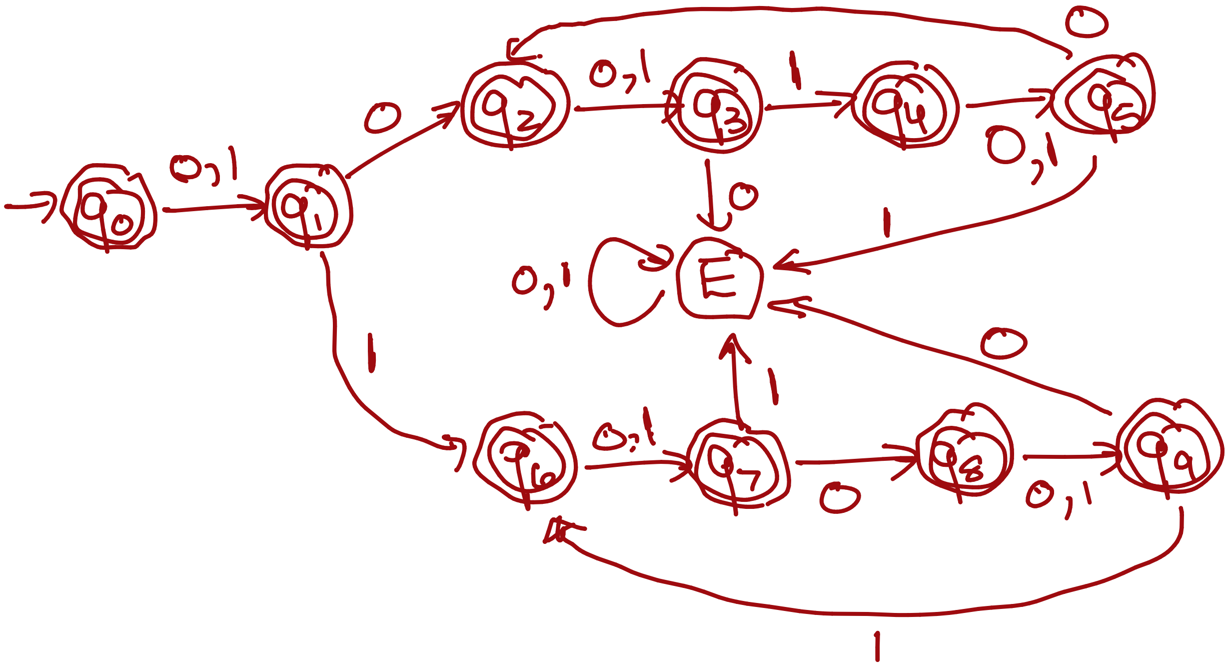
$$\leq \frac{1}{\left(\frac{64}{25}\right)},$$

so

$$\mathbb{P}[\text{device not damaged}] \geq 1 - \frac{25}{64}$$

$\blacksquare$

9 Give a DFA for strings whose even digits alternate between 0 and 1.



10 **For CFG $S \to 0S|S1|0|1$, prove that no string has 10 as a substring.**

Prf

We use induction on the length of the strings generated by the CFG (denoted len(s))

When len(s)=1, the strings cannot have 10 as a substring.

Assume no strings of length l derived from the CFG have 10 as a substring. Let s be a string of length $l+1$ derived from the CFG, then $s = 0w$ or $s = w1$ for a string w of length l derived from the CFG. By the inductive assumption w does not contain 10 as a substring. Since adding a 0 at the start, or a 1 at the end, of w does not introduce a 10 substring, s does not contain 10 as a substring.

We conclude by the principle of induction that no string derived from this CFG contains 10 as a substring.

11 **Consider the language of palindromes $\mathcal{L} = \{\omega \,|\, \omega \in \{0,1\}^*, \omega = \omega^R\}$. Give well-written high-level pseudocode for a decider for this language.**

1. Return to *

2. Go right to the first unmarked input. Mark and remember it. If there are no unmarked digits before you reach $\sqcup$, halt and ACCEPT

3. Go right to $\sqcup$, then go left to the first unmarked input. If it matches the remembered digit, mark it and GOTO 1. Otherwise, halt and REJECT.

12 **Given an ultimate-debugger $\mathcal{D}$ that takes $\langle M\rangle\#\omega$ and decides if TM M halts on input ω, show that every recognizer of a language $\mathcal{L}$ can be converted into a decider for the language.**

Let R be a recognizer for $\mathcal{L}$. Take Δ to be the Turing Machine that computes

$$\Delta(\omega) = \begin{cases} R(\omega) & \text{if } D(\langle R\rangle\#\omega) = \text{ACCEPT} \\ \text{REJECT} & \text{if } D(\langle R\rangle\#\omega) = \text{REJECT} \end{cases}$$

By construction,

1) If $\omega \in \mathcal{L}$, then $R(\omega)$ halts and ACCEPTS, so $D(\langle R\rangle\#\omega) = \text{ACCEPT}$, so
$$\Delta(\omega) = R(\omega) = \text{ACCEPT}$$

2) if $\omega \notin \mathcal{L}$, then either $R(\omega)$ does not halt, in which case $D(\langle R\rangle\#\omega) = \text{REJECT}$, so
$$\Delta(\omega) = \text{REJECT},$$
or $R(\omega)$ halts and REJECTS, so $D(\langle R\rangle\#\omega) = \text{ACCEPT}$, so
$$\Delta(\omega) = R(\omega) = \text{REJECT}.$$

That is,
$$\Delta(\omega) = \begin{cases} \text{ACCEPT} & \text{if } \omega \in \mathcal{L} \\ \text{REJECT} & \text{if } \omega \notin \mathcal{L} \end{cases}$$

so Δ is a decider for $\mathcal{L}$

$\blacksquare$

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