FINAL: 180 Minutes

Last Name: ______________________

First Name: ______________________

RIN: ______________________

Section: ______________________

Answer **ALL** questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

You **MUST** show **CORRECT** work to get full credit.

When in doubt, TINKER.

1	2	3	4	5	6	Total
200	30	30	30	30	30	350

1 Circle one answer per question. 10 points for each correct answer.

(1) If $P(3)$ is true and $P(n) \rightarrow (P(n^2) \wedge P(n-2))$ for $n \geq 2$, which of the following can I conclude?

A $P(120)$ is true

B $P(46)$ is not true

C $P(1)$ is not true

D $P(111)$ is true

E None of the above

(2) What is the most precise asymptotic relationship between $\sum_{i=1}^{n} i\sqrt{i}$ and n^2?

A $\sum_{i=1}^{n} i\sqrt{i} \in O(n^2)$

B $\sum_{i=1}^{n} i\sqrt{i} \in \Omega(n^2)$

C $\sum_{i=1}^{n} i\sqrt{i} \in \Theta(n^2)$

D $\sum_{i=1}^{n} i\sqrt{i} \in o(n^2)$

E $\sum_{i=1}^{n} i\sqrt{i} \in \omega(n^2)$

(3) In an ESP experiment, a pair of fair six-sided dice is rolled in a separate room and the supposed esper states that at least one of the dice came up six. What is the probability that the sum of the two dice is seven?

A 2/11

B 1/9

C 2/7

D 3/5

E None of the above

(4) What is the coefficient of x^{10} in $(x - x^{-1})^{20}$?

A $-\binom{20}{15}$

B $\binom{20}{5}$

C $-\binom{20}{10}$

D $\binom{20}{10}$

E None of the above

(5) What is the last digit of 3^{201}?

A 1

B 3

C 7

D 9

E None of the above

(6) Which of the following logical equivalences are correct?

(I) $p \vee q \to r \overset{\text{eqv}}{\equiv} \neg r \to \neg p \vee \neg q$
(II) $p \vee (q \wedge r) \overset{\text{eqv}}{\equiv} (p \wedge q) \vee (p \wedge r)$
(III) $(\neg q \vee r) \vee \neg p \overset{\text{eqv}}{\equiv} \neg q \vee (r \vee \neg p)$

A I

B II

C III

D I, II

E I, III

(7) Let the number of courses offered in the CSCI department be 27, and the number of courses offered in the math department be 25. If there are 5 courses cross-listed in both departments, how many different ways can we choose three distinct courses if those courses can be from either department?

A $\binom{25}{3}$

B $\binom{27}{3}$

C $\binom{47}{3}$

D $\binom{52}{3}$

E None of the above

(8) On a multiple choice test with 20 questions and 5 answers per question, how many different ways can the test be completed so that exactly half of the answers are wrong?

A $\binom{20}{10}\left(\frac{1}{5}\right)^{10}\left(\frac{4}{5}\right)^{10}$

B $\binom{20}{10}\left(\frac{4}{5}\right)^{10}$

C $\binom{20}{10}$

D $\binom{20}{10}10^4$

E $\binom{20}{10}4^{10}$

(9) Which set is *not* countable?

A The set of polynomials with rational coefficients and degree at most d.

B The set of circles in the plane with centers $c = (x, y)$ and radii r satisfying $x, y \in \mathbb{Z}$ and $r \in \mathbb{Q}$.

C The Cartesian product of two countable sets.

D The set of finite subsets of $\mathbb{Q}$.

E The set of irrational numbers.

(10) The graph G has six vertices with degrees $2, 2, 3, 4, 4, 5$. How many edges does G have?

A 8

B 10

C 12

D 20

E None of the above

(11) Which of these claims are true?

(I) If x and y are coprime, then $\mathbb{N} = \{mx + ny \mid m, n \in \mathbb{N}\}$.
(II) If x and y are coprime, then there is a $k \in \mathbb{N}$ for which $\gcd(xk, yk) \neq k$.

A I only.

B II only.

C Both I and II.

D Neither I or II.

E The truth values of these claims depend on the specific values of x and y.

(12) Which computing problem is context-free but not regular?

A $\mathcal{L} = \{\omega\#\omega \mid \omega \in \{0,1\}^*\}$

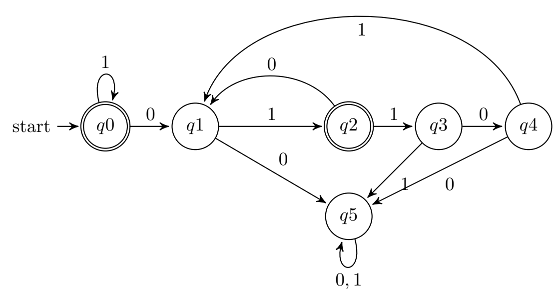
B $\mathcal{L} = \{\omega\#\omega^R \mid \omega \in \{0,1\}^*\}$

C $\mathcal{L} = \{1^{\bullet n}0^{\bullet n}1^{\bullet n} \mid n \geq 0\}$

D $\mathcal{L} = \{1^{\bullet(2n)}01^{\bullet(2k+1)} \mid n, k \geq 0\}$

E All of the above are regular languages

(13) Which string is accepted by this DFA?



A 1011

B 101000

C 101101

D 110011

E 1110101

(14) A team is three times more likely to win than lose each game it plays in a tournament. What is the expected number of games the team will play to get two wins?

A 8/3

B 3/2

C 2

D 3

E None of the above

(15) How many permutations of $\{1, \ldots, 7\}$ keep exactly four elements fixed (i.e. those elements are mapped to themselves under the permutation)? *Hint: equivalently, count the number of permutations that ensure that exactly three elements are not fixed.*

A 42

B 61

C 65

D 70

E None of the above

(16) A kindergarden teacher takes his class of ten children to the county fair and buys them cotton candy. The cotton candy machine has fifteen flavors. Each purchase yields a random flavor; each flavor has a non-zero probability of being selected. How many candies must the teacher buy to ensure all the children get the same flavor?

A 125

B 136

C 143

D 151

E None of the above

(17) Giselle has fifty wigs, and a display cabinet with six shelves. How many ways can Giselle arrange her wigs on these shelves if the order of the wigs on each shelf does not matter, and each shelf can hold all fifty wigs?

A 50^6

B 6^{50}

C $\binom{50}{6}$

D $50!/6!$

E None of the above

(18) Evaluate the sum $T(n) = \sum_{i=1}^{n} 2^{3-2i}$

$\boxed{\text{A}}$ $\frac{2}{3}\left(4 - 4^{-n}\right)$

$\boxed{\text{B}}$ $\frac{8}{3}\left(1 - 4^{-n}\right)$

$\boxed{\text{C}}$ $\frac{8}{3}\left(4 - 4^{-n}\right)$

$\boxed{\text{D}}$ $\frac{32}{3}\left(1 - 4^{-n}\right)$

$\boxed{\text{E}}$ $\frac{32}{3}\left(4 - 4^{-n}\right)$

(19) Which of the following is true?

(I) $\{0,1\}^*$ is decidable.
(II) If $\mathcal{L}$ is decidable and $\mathcal{S} \subseteq \mathcal{L}$, then $\mathcal{S}$ is decidable.
(III) If $\mathcal{L}$ is recognizable and $\mathcal{S} \subseteq \mathcal{L}$, then $\mathcal{S}$ is recognizable.

$\boxed{\text{A}}$ I only

$\boxed{\text{B}}$ II only

$\boxed{\text{C}}$ III only

$\boxed{\text{D}}$ I and III only

$\boxed{\text{E}}$ I, II, and III

(20) Which of these degree sequences is graphical? (Recall that a degree sequence is graphical iff there exists a graph with that degree sequence)

$\boxed{\text{A}}$ $[3, 2, 2, 2]$

$\boxed{\text{B}}$ $[3, 3, 3, 1]$

$\boxed{\text{C}}$ $[3, 3, 3, 2]$

$\boxed{\text{D}}$ $[3, 3, 3, 3]$

$\boxed{\text{E}}$ None of the above

2 **Let $\mathcal{V}$ be a set of n vertices, and let the edge set $\mathcal{E}$ be initially empty. For each pair of vertices $i \neq j$, add the edge (i, j) to $\mathcal{E}$ with probability p. Let $\mathbf{X} = |\mathcal{E}|$ be the number of edges. Compute the expectation and variance of $\mathbf{X}$.**

3 **Show that the recognizable languages are closed under intersection: let $\mathcal{L}_1$ and $\mathcal{L}_2$ be recognizable, and show that $\mathcal{L}_\cap = \mathcal{L}_1 \cap \mathcal{L}_2$ is recognizable by using the recognizers for $\mathcal{L}_1$ and $\mathcal{L}_2$ to construct a recognizer for $\mathcal{L}_\cap$. Prove that it is indeed a recognizer.**

4 **Consider the language of even-length palindromes $\mathcal{L} = \{\omega\omega^R \,|\, \omega \in \{0,1\}^*\}$. Give well-written pseudocode for a decider for this language.**

5 **Consider the following process of assigning final projects to k students. Each student is assigned their final project uniformly at random from n possible projects ($n \geq k$), independent of the other students' assignments. If each student is assigned a unique project, then the project selection process finishes. If any two students are assigned the same project, the project selection process restarts. What is the expected number of times this project selection process will be used before the students all have unique projects?**

6 **Solve the recurrence relation** $T(n) = 2T(n-1) - T(n-2) + 2$ **when** $T(0) = 1$ **and** $T(1) = 2$**.**

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