Foundations of Computer Science
Lecture 22

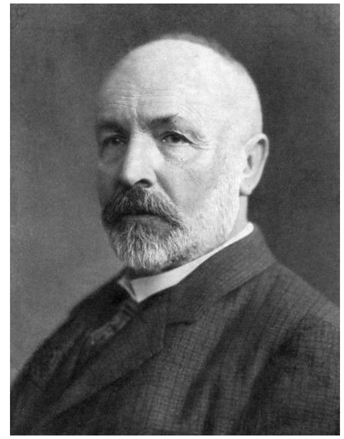
Infinity

Size versus Cardinality: Comparing "Sizes"
Countable: Sets Which Are Not "Larger" Than ℕ
Is There A Set "Larger" Than ℕ? Cantor's Diagonal Argument
Infinity and Computing

Our Short Stroll Through Discrete Math

1. Precise statements, proofs and logic.
2. **INDUCTION**.
3. Recursively defined structures and Induction. (Data structures; PL)
4. Sums and asymptotics. (Algorithm analysis)
5. Number theory. (Cryptography; probability; fun)
6. Graphs. (Relationships/conflicts; resource allocation; routing; scheduling,. . .)
7. Counting. (Enumeration and brute force algorithms)
8. Probability. (Real world algorithms involve randomness/uncertainty)
   - Inputs arrive in a random order;
   - Randomized algorithms (primality testing, machine learning, routing, conflict resolution . . .)
   - Expected value is a summary of what happens. Variance tells you how good the summary is.

Today: Infinity



Georg Cantor

1. Comparing "sizes" of sets: countable.
   - Rationals are countable.

2. Uncountable
   - Infinite binary strings.

3. What does Infinity have to do with computing?

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You must know how to count.

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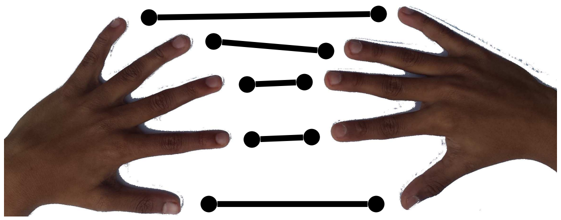

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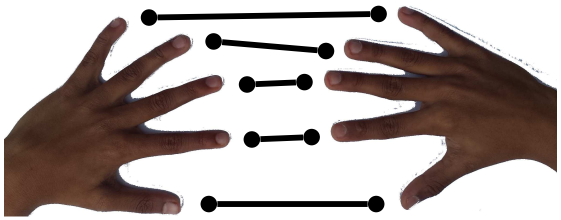

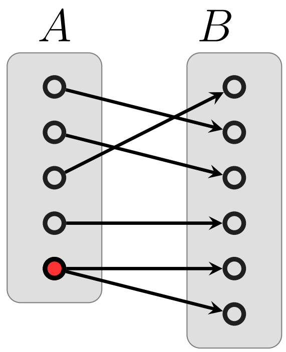

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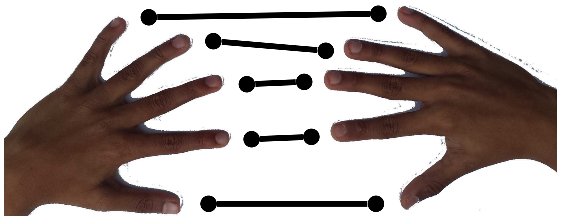

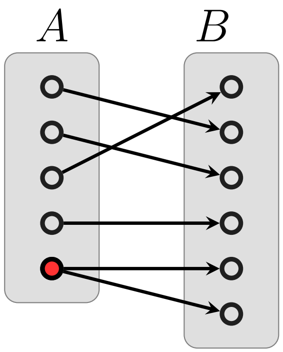

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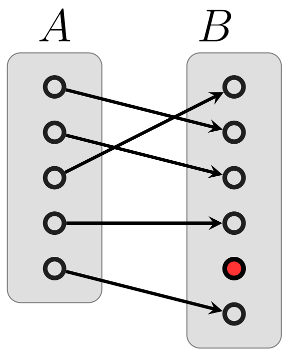

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(injection, $A \overset{\text{INJ}}{\mapsto} B$)
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Cardinality $|A|$(“size”), read “cardinality of A,” is the number of elements for finite sets

$|A| \leq |B|$ iff there is an injection (1-to-1) from A to B, i.e., $f : A \overset{\text{INJ}}{\mapsto} B$.
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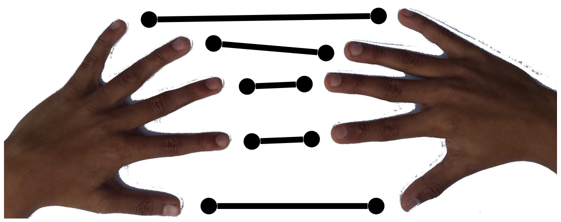

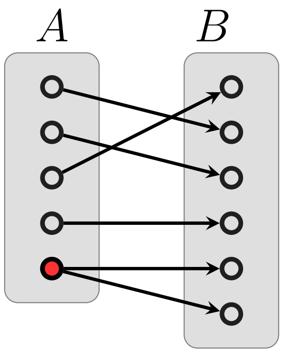

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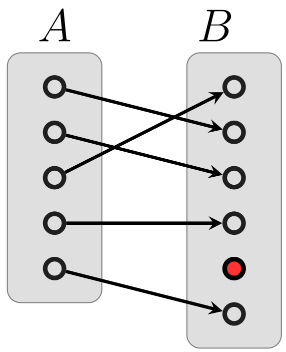

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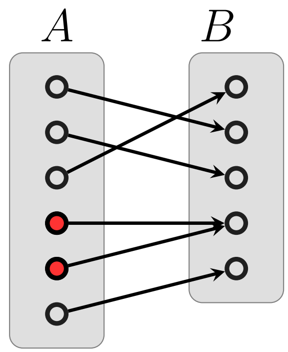

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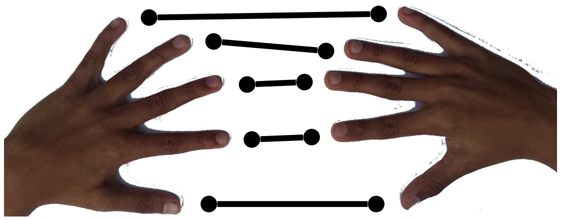

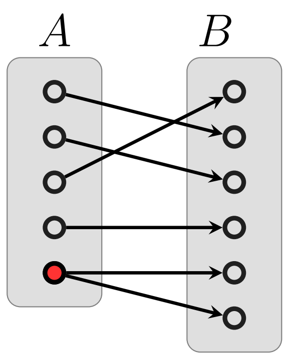

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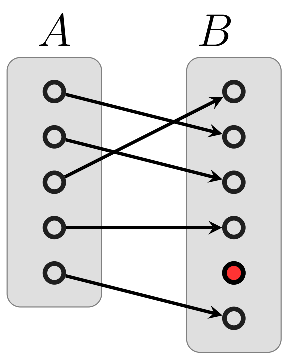

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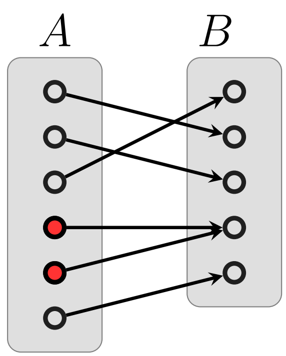

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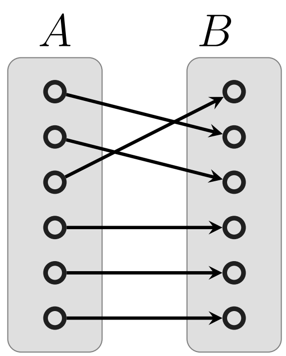

1-to-1 **and** onto
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$|A| \geq |B|$ iff there is an surjection (onto) from A to B, i.e., $f : A \overset{\text{SUR}}{\mapsto} B$.

$|A| = |B|$ iff there is an bijection (1-to-1 and onto) from A to B, i.e., $f : A \overset{\text{BIJ}}{\mapsto} B$.

$$|A| \leq |B| \text{ AND } |B| \leq |A| \rightarrow |A| = |B|. \qquad \text{(Cantor-Bernstein Theorem)}$$

A Countable Set's Cardinality Is At Most $|\mathbb{N}|$

Finite sets : $|A| = n$ if and only if there is a bijection from A to $\{1, \ldots, n\}$.

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A :	⊃	✒	❖	✠	✚	✘	⧫	✣	❉	▲	❒	◗	❍	$\cdots$						
$\mathbb{N}$:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20 $\cdots$

You cannot skip over any elements of A, but you might not use every element of $\mathbb{N}$.

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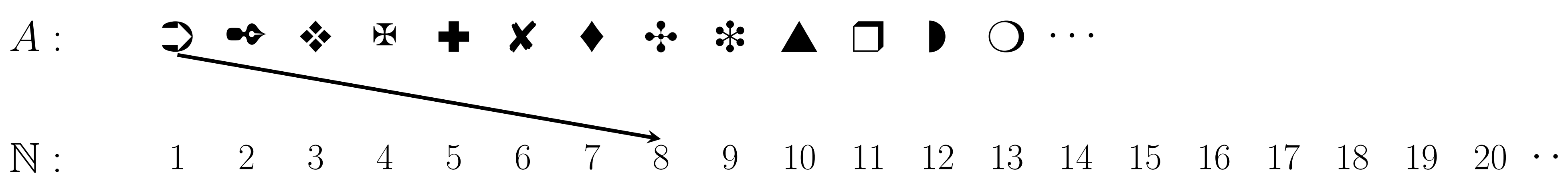

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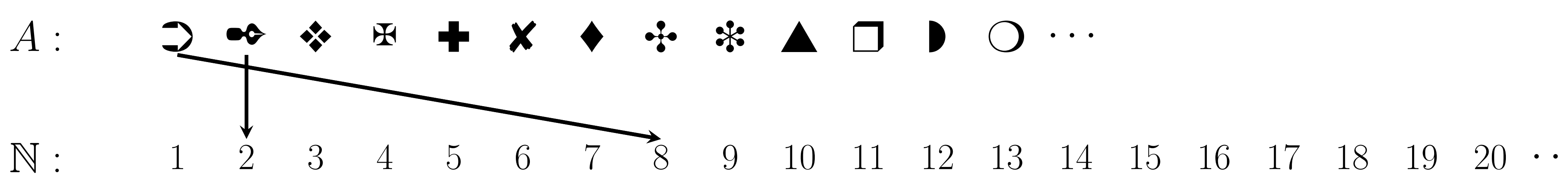

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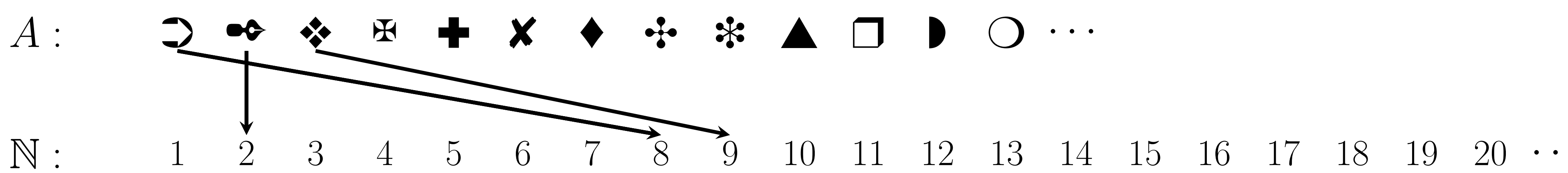

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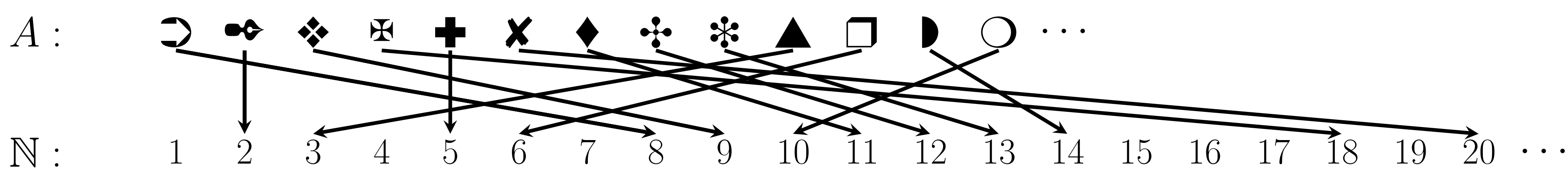

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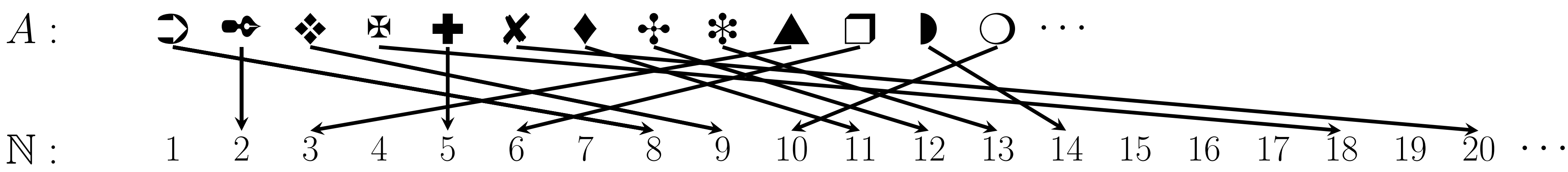

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<u>To prove that a function $f : A \mapsto \mathbb{N}$ is an injection:</u>

1: Assume f is *not* an injection. (Proof by contradiction.)
2: This means there is a pair $x, y \in A$ for which $x \neq y$ and $f(x) = f(y)$.
3: Use $f(x) = f(y)$ to prove that $x = y$, a contradiction. Hence, f *is* an injection.

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$$3 \mapsto 1 \qquad 6 \mapsto 2 \qquad 8 \mapsto 3.$$

For an arbitrary finite set $A = \{a_1, a_2, \ldots, a_n\}$, $\mathbb{N}$,

$$a_1 \mapsto 1 \qquad a_2 \mapsto 2 \qquad a_3 \mapsto 3 \qquad \cdots \qquad a_n \mapsto n.$$

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To prove $|\,\mathbb{N}_0\,| \le |\,\mathbb{N}\,|$, we give an injection $f : \mathbb{N}_0 \overset{\text{INJ}}{\mapsto} \mathbb{N}$,

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Also, $|\,\mathbb{N}\,| \le |\,\mathbb{N}_0\,|$ because $\mathbb{N} \subseteq \mathbb{N}_0 \to |\,\mathbb{N}_0\,| = |\,\mathbb{N}\,|$. (Cantor-Bernstein)

Bijection:

$\mathbb{N}_0$:	0	1	2	3	4	5	6	7	8	9	$\cdots$
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{N}$:	1	2	3	4	5	6	7	8	9	10	$\cdots$

Positive Even Numbers and Integers are Countable

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$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. $|\mathbb{Z}| = |\mathbb{N}|$.

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	0		-1		-2		-3		-4		-5		-6		-7		-8		-9		-10 $\cdots$
	$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$
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		1		2		3		4		5		6		7		8		9		10 $\cdots$	

Exercise. What is a mathematical formula for the bijection?

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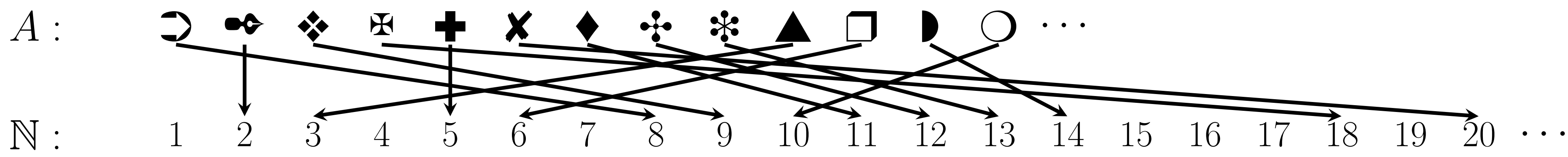

Every Countable Set Can Be "Listed"

$\{3, 6, 8\}$ is a list. $E = \{2, 4, 6, \ldots\}$ is a list. What about $\mathbb{Z}$?

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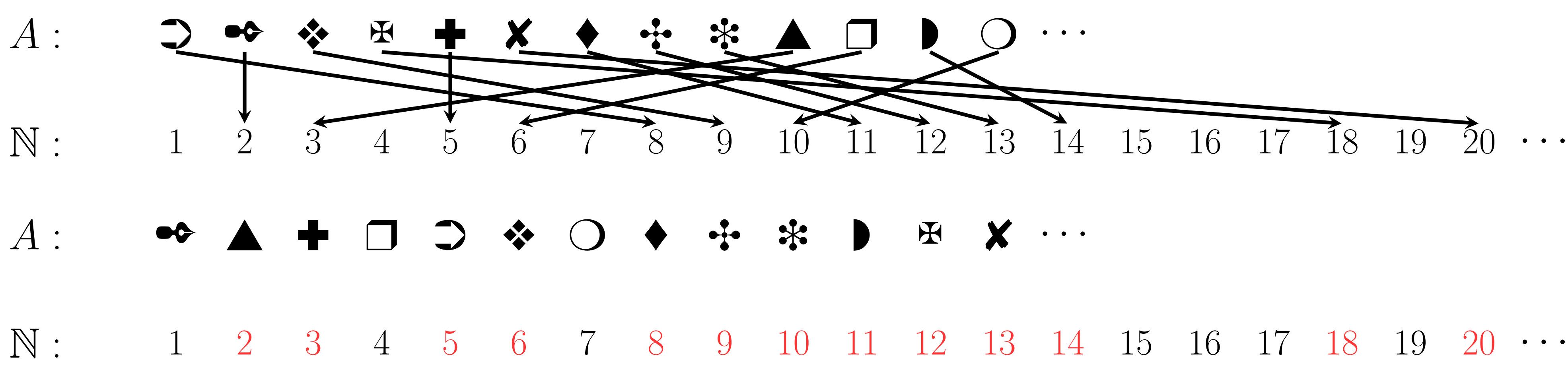

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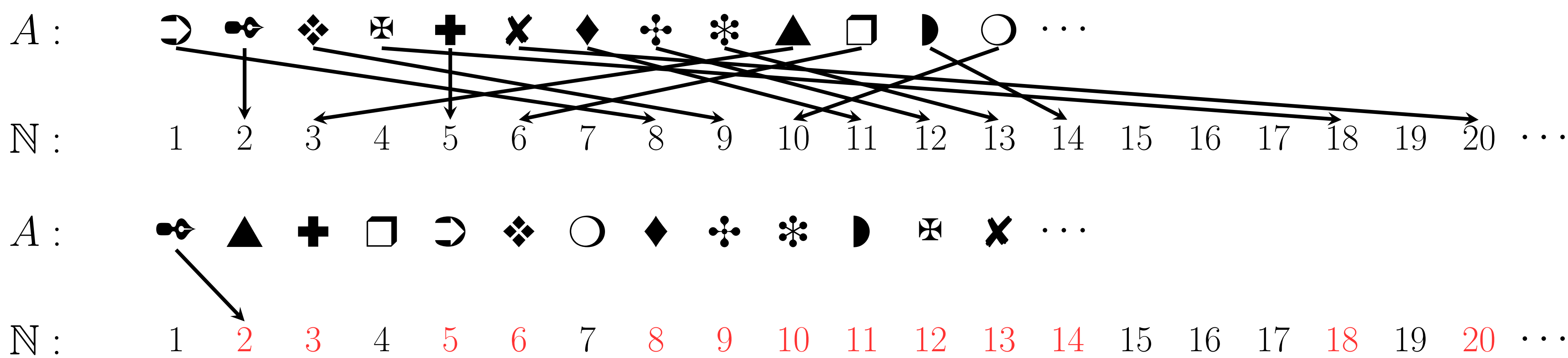

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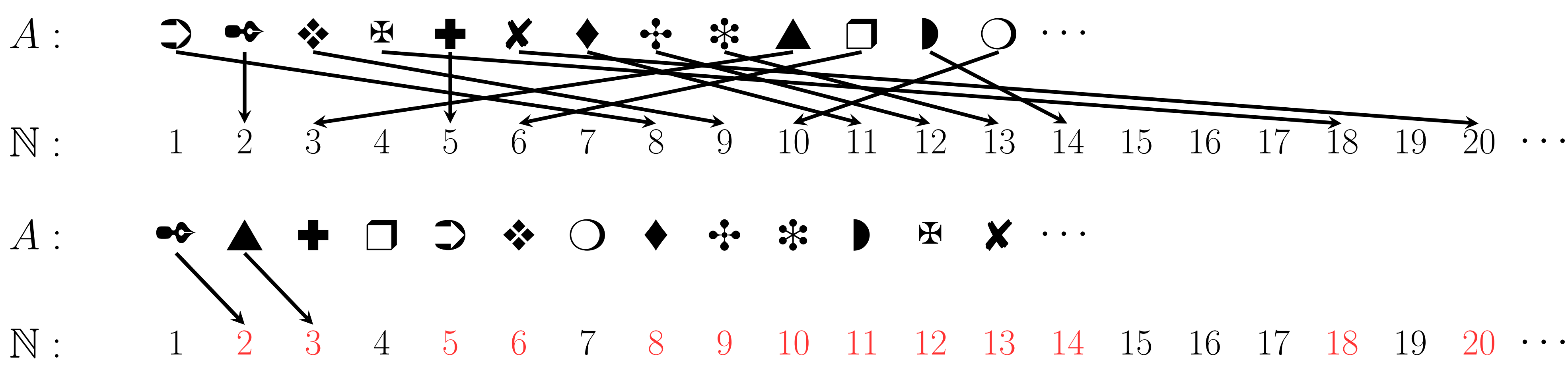

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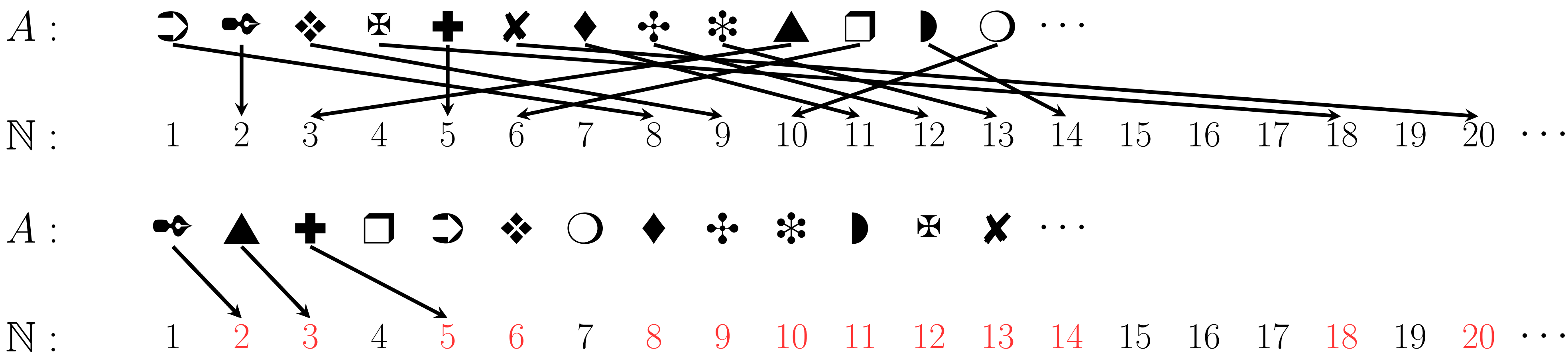

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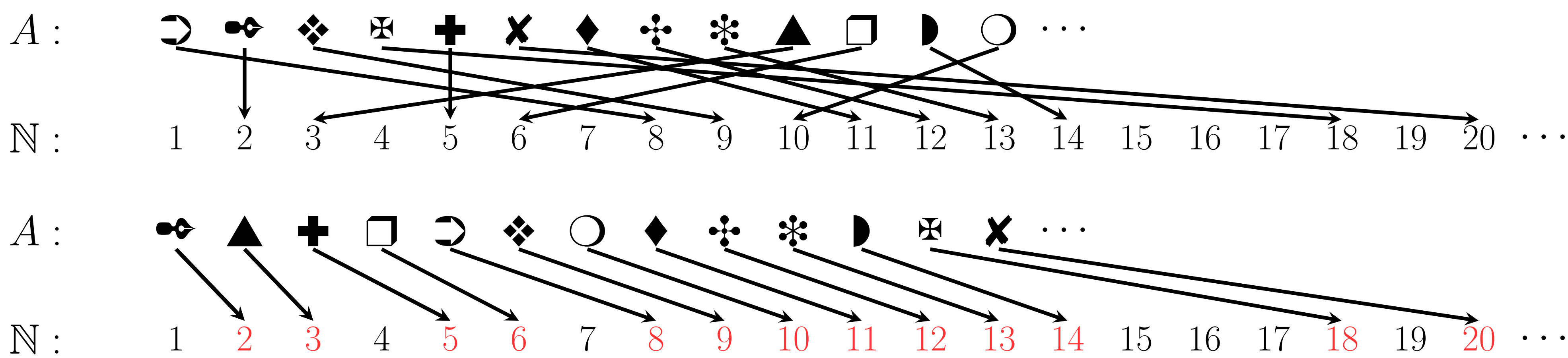

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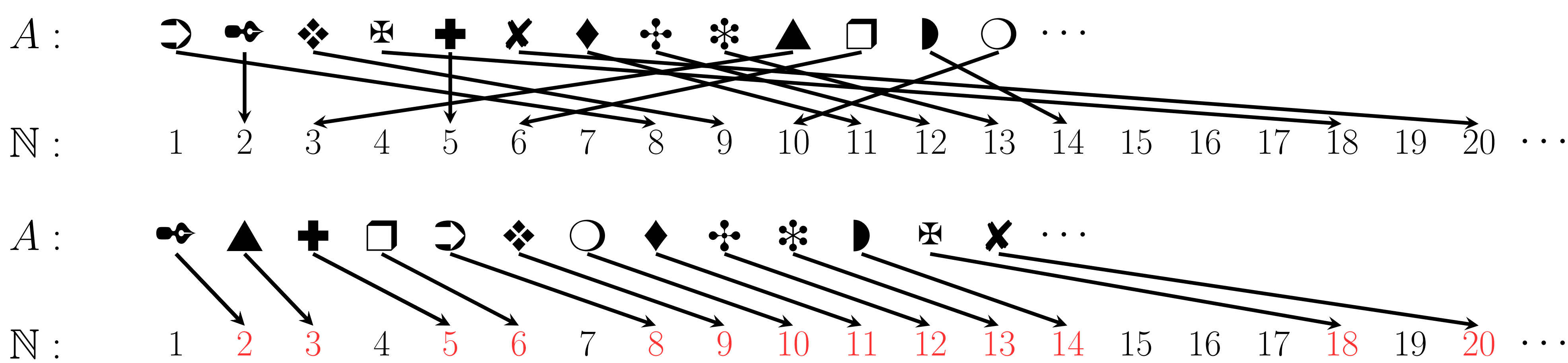

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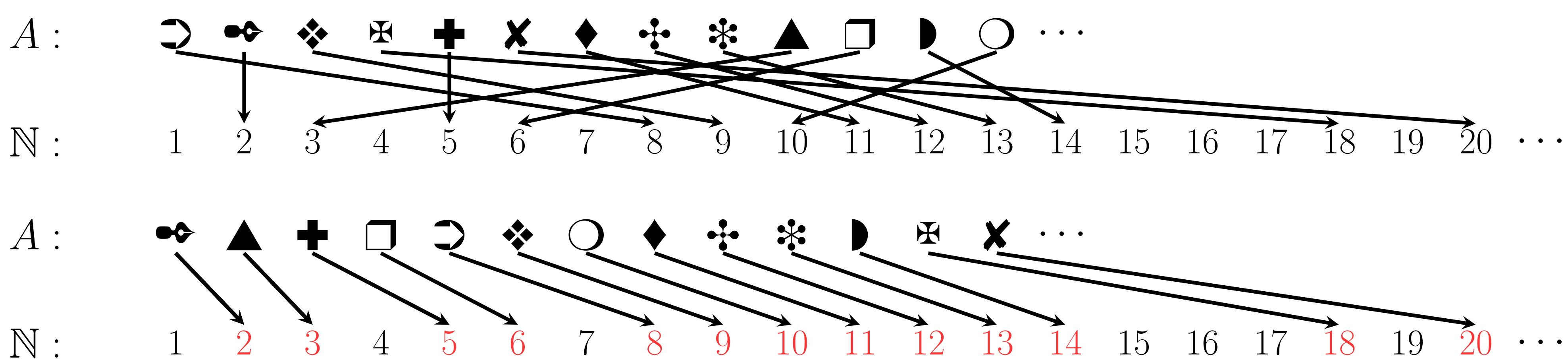

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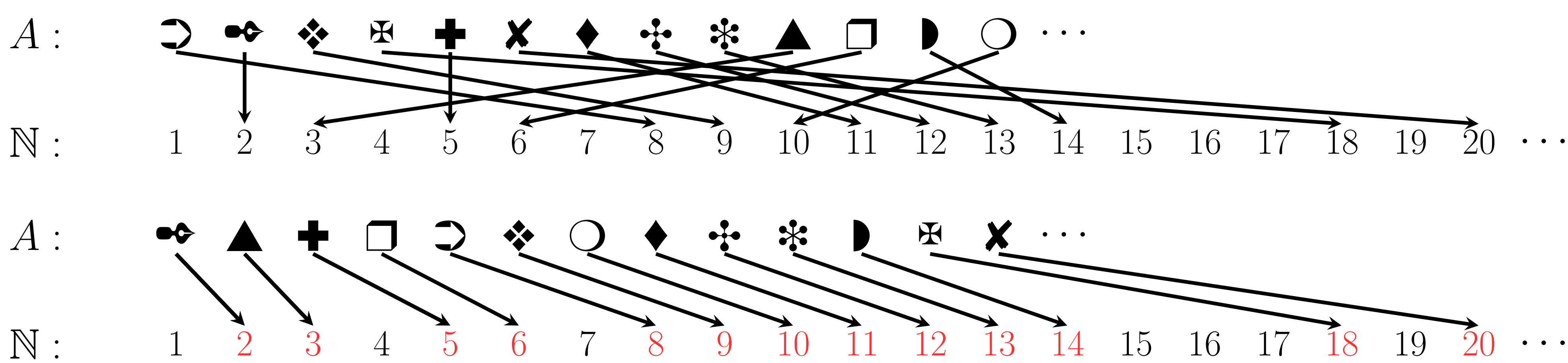

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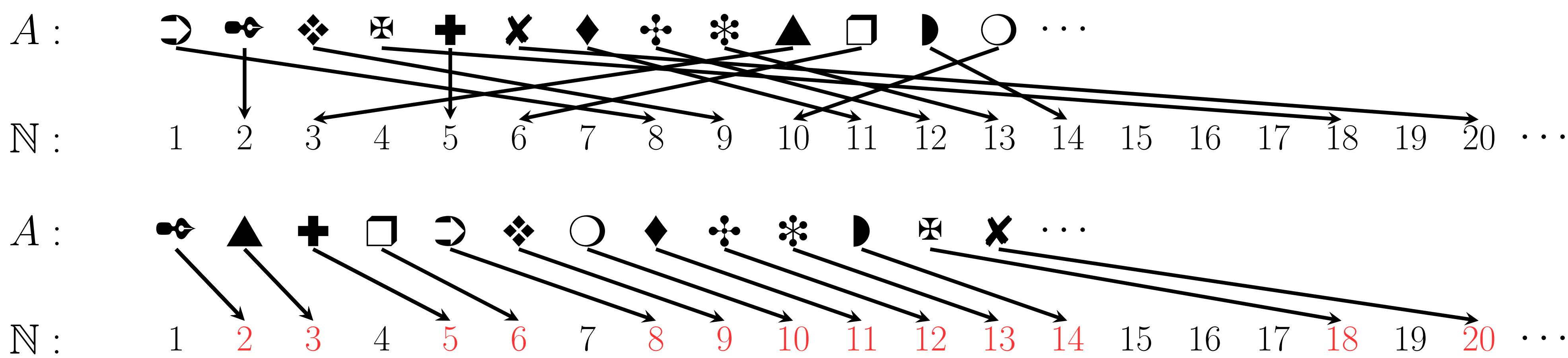

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$$\text{list position of } z = \begin{cases} 2z & z > 0; \\ 2|z| + 1 & z \leq 0; \end{cases}$$

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A and B are countable, so they can be listed.

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list-position of a_i is $2i - 1$;
list-position of b_i is $2i$.

Pop Quiz. Get a list of $\mathbb{Z}$ with $A = \{0, -1, -2, -3, \ldots\}$ and $B = \{1, 2, 3, \ldots\}$ using union.

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1	$\frac{0}{1}$	$\frac{+1}{1}$	$\frac{-1}{1}$	$\frac{+2}{1}$	$\frac{-2}{1}$	$\frac{+3}{1}$	$\frac{-3}{1}$	$\frac{+4}{1}$	$\frac{-4}{1}$	$\cdots$
2	$\frac{0}{2}$	$\frac{+1}{2}$	$\frac{-1}{2}$	$\frac{+2}{2}$	$\frac{-2}{2}$	$\frac{+3}{2}$	$\frac{-3}{2}$	$\frac{+4}{2}$	$\frac{-4}{2}$	$\cdots$
$\mathbb{N}$ 3	$\frac{0}{3}$	$\frac{+1}{3}$	$\frac{-1}{3}$	$\frac{+2}{3}$	$\frac{-2}{3}$	$\frac{+3}{3}$	$\frac{-3}{3}$	$\frac{+4}{3}$	$\frac{-4}{3}$	$\cdots$
4	$\frac{0}{4}$	$\frac{+1}{4}$	$\frac{-1}{4}$	$\frac{+2}{4}$	$\frac{-2}{4}$	$\frac{+3}{4}$	$\frac{-3}{4}$	$\frac{+4}{4}$	$\frac{-4}{4}$	$\cdots$
5	$\frac{0}{5}$	$\frac{+1}{5}$	$\frac{-1}{5}$	$\frac{+2}{5}$	$\frac{-2}{5}$	$\frac{+3}{5}$	$\frac{-3}{5}$	$\frac{+4}{5}$	$\frac{-4}{5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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$|\mathbb{Q}| = |\mathbb{N}| \times |\mathbb{Z}| \gg |\mathbb{N}|$. ✘ ☹

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$\mathbb{Q}$	$\mathbb{Z}$: 0	$+1$	-1	$+2$	-2	$+3$	-3	$+4$	-4	$\cdots$
1	$\frac{0}{1}$	$\frac{+1}{1}$	$\frac{-1}{1}$	$\frac{+2}{1}$	$\frac{-2}{1}$	$\frac{+3}{1}$	$\frac{-3}{1}$	$\frac{+4}{1}$	$\frac{-4}{1}$	$\cdots$
2	$\frac{0}{2}$	$\frac{+1}{2}$	$\frac{-1}{2}$	$\frac{+2}{2}$	$\frac{-2}{2}$	$\frac{+3}{2}$	$\frac{-3}{2}$	$\frac{+4}{2}$	$\frac{-4}{2}$	$\cdots$
$\mathbb{N}$ 3	$\frac{0}{3}$	$\frac{+1}{3}$	$\frac{-1}{3}$	$\frac{+2}{3}$	$\frac{-2}{3}$	$\frac{+3}{3}$	$\frac{-3}{3}$	$\frac{+4}{3}$	$\frac{-4}{3}$	$\cdots$
4	$\frac{0}{4}$	$\frac{+1}{4}$	$\frac{-1}{4}$	$\frac{+2}{4}$	$\frac{-2}{4}$	$\frac{+3}{4}$	$\frac{-3}{4}$	$\frac{+4}{4}$	$\frac{-4}{4}$	$\cdots$
5	$\frac{0}{5}$	$\frac{+1}{5}$	$\frac{-1}{5}$	$\frac{+2}{5}$	$\frac{-2}{5}$	$\frac{+3}{5}$	$\frac{-3}{5}$	$\frac{+4}{5}$	$\frac{-4}{5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Intuition suggests
$|\,\mathbb{Q}\,| = |\,\mathbb{N}\,| \times |\,\mathbb{Z}\,| \gg |\,\mathbb{N}\,|$. ✘ ☹

Rationals are Countable: $|\mathbb{Q}| = |\mathbb{N}|$

This is surprising because between any two rationals there is another (not true for $\mathbb{N}$).

$\mathbb{Q}$	$\mathbb{Z}$: 0	$+1$	-1	$+2$	-2	$+3$	-3	$+4$	-4	$\cdots$
1	$\frac{0}{1}$	$\frac{+1}{1}$	$\frac{-1}{1}$	$\frac{+2}{1}$	$\frac{-2}{1}$	$\frac{+3}{1}$	$\frac{-3}{1}$	$\frac{+4}{1}$	$\frac{-4}{1}$	$\cdots$
2	$\frac{0}{2}$	$\frac{+1}{2}$	$\frac{-1}{2}$	$\frac{+2}{2}$	$\frac{-2}{2}$	$\frac{+3}{2}$	$\frac{-3}{2}$	$\frac{+4}{2}$	$\frac{-4}{2}$	$\cdots$
$\mathbb{N}$ 3	$\frac{0}{3}$	$\frac{+1}{3}$	$\frac{-1}{3}$	$\frac{+2}{3}$	$\frac{-2}{3}$	$\frac{+3}{3}$	$\frac{-3}{3}$	$\frac{+4}{3}$	$\frac{-4}{3}$	$\cdots$
4	$\frac{0}{4}$	$\frac{+1}{4}$	$\frac{-1}{4}$	$\frac{+2}{4}$	$\frac{-2}{4}$	$\frac{+3}{4}$	$\frac{-3}{4}$	$\frac{+4}{4}$	$\frac{-4}{4}$	$\cdots$
5	$\frac{0}{5}$	$\frac{+1}{5}$	$\frac{-1}{5}$	$\frac{+2}{5}$	$\frac{-2}{5}$	$\frac{+3}{5}$	$\frac{-3}{5}$	$\frac{+4}{5}$	$\frac{-4}{5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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$\mathbb{Q}$					$\mathbb{Z}$					
	0	$+1$	-1	$+2$	-2	$+3$	-3	$+4$	-4	$\cdots$
1	$\frac{0}{1}$	$\frac{+1}{1}$	$\frac{-1}{1}$	$\frac{+2}{1}$	$\frac{-2}{1}$	$\frac{+3}{1}$	$\frac{-3}{1}$	$\frac{+4}{1}$	$\frac{-4}{1}$	$\cdots$
2	$\frac{0}{2}$	$\frac{+1}{2}$	$\frac{-1}{2}$	$\frac{+2}{2}$	$\frac{-2}{2}$	$\frac{+3}{2}$	$\frac{-3}{2}$	$\frac{+4}{2}$	$\frac{-4}{2}$	$\cdots$
$\mathbb{N}$ 3	$\frac{0}{3}$	$\frac{+1}{3}$	$\frac{-1}{3}$	$\frac{+2}{3}$	$\frac{-2}{3}$	$\frac{+3}{3}$	$\frac{-3}{3}$	$\frac{+4}{3}$	$\frac{-4}{3}$	$\cdots$
4	$\frac{0}{4}$	$\frac{+1}{4}$	$\frac{-1}{4}$	$\frac{+2}{4}$	$\frac{-2}{4}$	$\frac{+3}{4}$	$\frac{-3}{4}$	$\frac{+4}{4}$	$\frac{-4}{4}$	$\cdots$
5	$\frac{0}{5}$	$\frac{+1}{5}$	$\frac{-1}{5}$	$\frac{+2}{5}$	$\frac{-2}{5}$	$\frac{+3}{5}$	$\frac{-3}{5}$	$\frac{+4}{5}$	$\frac{-4}{5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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$\mathbb{Q}$		$\mathbb{Z}$								
	0	$+1$	-1	$+2$	-2	$+3$	-3	$+4$	-4	$\cdots$
1	$\frac{0}{1}$	$\frac{+1}{1}$	$\frac{-1}{1}$	$\frac{+2}{1}$	$\frac{-2}{1}$	$\frac{+3}{1}$	$\frac{-3}{1}$	$\frac{+4}{1}$	$\frac{-4}{1}$	$\cdots$
2	$\frac{0}{2}$	$\frac{+1}{2}$	$\frac{-1}{2}$	$\frac{+2}{2}$	$\frac{-2}{2}$	$\frac{+3}{2}$	$\frac{-3}{2}$	$\frac{+4}{2}$	$\frac{-4}{2}$	$\cdots$
$\mathbb{N}$ 3	$\frac{0}{3}$	$\frac{+1}{3}$	$\frac{-1}{3}$	$\frac{+2}{3}$	$\frac{-2}{3}$	$\frac{+3}{3}$	$\frac{-3}{3}$	$\frac{+4}{3}$	$\frac{-4}{3}$	$\cdots$
4	$\frac{0}{4}$	$\frac{+1}{4}$	$\frac{-1}{4}$	$\frac{+2}{4}$	$\frac{-2}{4}$	$\frac{+3}{4}$	$\frac{-3}{4}$	$\frac{+4}{4}$	$\frac{-4}{4}$	$\cdots$
5	$\frac{0}{5}$	$\frac{+1}{5}$	$\frac{-1}{5}$	$\frac{+2}{5}$	$\frac{-2}{5}$	$\frac{+3}{5}$	$\frac{-3}{5}$	$\frac{+4}{5}$	$\frac{-4}{5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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$\mathbb{Q}$	$\mathbb{Z}$: 0	+1	−1	+2	−2	+3	−3	+4	−4	$\cdots$
1	$\frac{0}{1}$	$\frac{+1}{1}$	$\frac{-1}{1}$	$\frac{+2}{1}$	$\frac{-2}{1}$	$\frac{+3}{1}$	$\frac{-3}{1}$	$\frac{+4}{1}$	$\frac{-4}{1}$	$\cdots$
2	$\frac{0}{2}$	$\frac{+1}{2}$	$\frac{-1}{2}$	$\frac{+2}{2}$	$\frac{-2}{2}$	$\frac{+3}{2}$	$\frac{-3}{2}$	$\frac{+4}{2}$	$\frac{-4}{2}$	$\cdots$
$\mathbb{N}$ 3	$\frac{0}{3}$	$\frac{+1}{3}$	$\frac{-1}{3}$	$\frac{+2}{3}$	$\frac{-2}{3}$	$\frac{+3}{3}$	$\frac{-3}{3}$	$\frac{+4}{3}$	$\frac{-4}{3}$	$\cdots$
4	$\frac{0}{4}$	$\frac{+1}{4}$	$\frac{-1}{4}$	$\frac{+2}{4}$	$\frac{-2}{4}$	$\frac{+3}{4}$	$\frac{-3}{4}$	$\frac{+4}{4}$	$\frac{-4}{4}$	$\cdots$
5	$\frac{0}{5}$	$\frac{+1}{5}$	$\frac{-1}{5}$	$\frac{+2}{5}$	$\frac{-2}{5}$	$\frac{+3}{5}$	$\frac{-3}{5}$	$\frac{+4}{5}$	$\frac{-4}{5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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$\mathbb{Q}$		$\mathbb{Z}$: 0	$+1$	-1	$+2$	-2	$+3$	-3	$+4$	-4	$\cdots$
	1	$\frac{0}{1}$	$\frac{+1}{1}$	$\frac{-1}{1}$	$\frac{+2}{1}$	$\frac{-2}{1}$	$\frac{+3}{1}$	$\frac{-3}{1}$	$\frac{+4}{1}$	$\frac{-4}{1}$	$\cdots$
	2	$\frac{0}{2}$	$\frac{+1}{2}$	$\frac{-1}{2}$	$\frac{+2}{2}$	$\frac{-2}{2}$	$\frac{+3}{2}$	$\frac{-3}{2}$	$\frac{+4}{2}$	$\frac{-4}{2}$	$\cdots$
$\mathbb{N}$	3	$\frac{0}{3}$	$\frac{+1}{3}$	$\frac{-1}{3}$	$\frac{+2}{3}$	$\frac{-2}{3}$	$\frac{+3}{3}$	$\frac{-3}{3}$	$\frac{+4}{3}$	$\frac{-4}{3}$	$\cdots$
	4	$\frac{0}{4}$	$\frac{+1}{4}$	$\frac{-1}{4}$	$\frac{+2}{4}$	$\frac{-2}{4}$	$\frac{+3}{4}$	$\frac{-3}{4}$	$\frac{+4}{4}$	$\frac{-4}{4}$	$\cdots$
	5	$\frac{0}{5}$	$\frac{+1}{5}$	$\frac{-1}{5}$	$\frac{+2}{5}$	$\frac{-2}{5}$	$\frac{+3}{5}$	$\frac{-3}{5}$	$\frac{+4}{5}$	$\frac{-4}{5}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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$\mathbb{Q}$		$\mathbb{Z}$: 0	$+1$	-1	$+2$	-2	$+3$	-3	$+4$	-4	$\cdots$
	1	$\frac{0}{1}$	$\frac{+1}{1}$	$\frac{-1}{1}$	$\frac{+2}{1}$	$\frac{-2}{1}$	$\frac{+3}{1}$	$\frac{-3}{1}$	$\frac{+4}{1}$	$\frac{-4}{1}$	$\cdots$
	2	$\frac{0}{2}$	$\frac{+1}{2}$	$\frac{-1}{2}$	$\frac{+2}{2}$	$\frac{-2}{2}$	$\frac{+3}{2}$	$\frac{-3}{2}$	$\frac{+4}{2}$	$\frac{-4}{2}$	$\cdots$
$\mathbb{N}$	3	$\frac{0}{3}$	$\frac{+1}{3}$	$\frac{-1}{3}$	$\frac{+2}{3}$	$\frac{-2}{3}$	$\frac{+3}{3}$	$\frac{-3}{3}$	$\frac{+4}{3}$	$\frac{-4}{3}$	$\cdots$
	4	$\frac{0}{4}$	$\frac{+1}{4}$	$\frac{-1}{4}$	$\frac{+2}{4}$	$\frac{-2}{4}$	$\frac{+3}{4}$	$\frac{-3}{4}$	$\frac{+4}{4}$	$\frac{-4}{4}$	$\cdots$
	5	$\frac{0}{5}$	$\frac{+1}{5}$	$\frac{-1}{5}$	$\frac{+2}{5}$	$\frac{-2}{5}$	$\frac{+3}{5}$	$\frac{-3}{5}$	$\frac{+4}{5}$	$\frac{-4}{5}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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$\mathbb{Q}$		$\mathbb{Z}$: 0	$+1$	-1	$+2$	-2	$+3$	-3	$+4$	-4	$\cdots$
	1	$\frac{0}{1}$	$\frac{+1}{1}$	$\frac{-1}{1}$	$\frac{+2}{1}$	$\frac{-2}{1}$	$\frac{+3}{1}$	$\frac{-3}{1}$	$\frac{+4}{1}$	$\frac{-4}{1}$	$\cdots$
	2	$\frac{0}{2}$	$\frac{+1}{2}$	$\frac{-1}{2}$	$\frac{+2}{2}$	$\frac{-2}{2}$	$\frac{+3}{2}$	$\frac{-3}{2}$	$\frac{+4}{2}$	$\frac{-4}{2}$	$\cdots$
$\mathbb{N}$	3	$\frac{0}{3}$	$\frac{+1}{3}$	$\frac{-1}{3}$	$\frac{+2}{3}$	$\frac{-2}{3}$	$\frac{+3}{3}$	$\frac{-3}{3}$	$\frac{+4}{3}$	$\frac{-4}{3}$	$\cdots$
	4	$\frac{0}{4}$	$\frac{+1}{4}$	$\frac{-1}{4}$	$\frac{+2}{4}$	$\frac{-2}{4}$	$\frac{+3}{4}$	$\frac{-3}{4}$	$\frac{+4}{4}$	$\frac{-4}{4}$	$\cdots$
	5	$\frac{0}{5}$	$\frac{+1}{5}$	$\frac{-1}{5}$	$\frac{+2}{5}$	$\frac{-2}{5}$	$\frac{+3}{5}$	$\frac{-3}{5}$	$\frac{+4}{5}$	$\frac{-4}{5}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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$\mathbb{Q}$	$\mathbb{Z}$: 0	$+1$	-1	$+2$	-2	$+3$	-3	$+4$	-4	$\cdots$
$\mathbb{N}$: 1	$\frac{0}{1}$	$\frac{+1}{1}$	$\frac{-1}{1}$	$\frac{+2}{1}$	$\frac{-2}{1}$	$\frac{+3}{1}$	$\frac{-3}{1}$	$\frac{+4}{1}$	$\frac{-4}{1}$	$\cdots$
2	$\frac{0}{2}$	$\frac{+1}{2}$	$\frac{-1}{2}$	$\frac{+2}{2}$	$\frac{-2}{2}$	$\frac{+3}{2}$	$\frac{-3}{2}$	$\frac{+4}{2}$	$\frac{-4}{2}$	$\cdots$
3	$\frac{0}{3}$	$\frac{+1}{3}$	$\frac{-1}{3}$	$\frac{+2}{3}$	$\frac{-2}{3}$	$\frac{+3}{3}$	$\frac{-3}{3}$	$\frac{+4}{3}$	$\frac{-4}{3}$	$\cdots$
4	$\frac{0}{4}$	$\frac{+1}{4}$	$\frac{-1}{4}$	$\frac{+2}{4}$	$\frac{-2}{4}$	$\frac{+3}{4}$	$\frac{-3}{4}$	$\frac{+4}{4}$	$\frac{-4}{4}$	$\cdots$
5	$\frac{0}{5}$	$\frac{+1}{5}$	$\frac{-1}{5}$	$\frac{+2}{5}$	$\frac{-2}{5}$	$\frac{+3}{5}$	$\frac{-3}{5}$	$\frac{+4}{5}$	$\frac{-4}{5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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$\mathbb{Q}$ \ $\mathbb{Z}$	0	+1	−1	+2	−2	+3	−3	+4	−4	⋯
$\mathbb{N}$ 1	$\frac{0}{1}$	$\frac{+1}{1}$	$\frac{-1}{1}$	$\frac{+2}{1}$	$\frac{-2}{1}$	$\frac{+3}{1}$	$\frac{-3}{1}$	$\frac{+4}{1}$	$\frac{-4}{1}$	⋯
2	$\frac{0}{2}$	$\frac{+1}{2}$	$\frac{-1}{2}$	$\frac{+2}{2}$	$\frac{-2}{2}$	$\frac{+3}{2}$	$\frac{-3}{2}$	$\frac{+4}{2}$	$\frac{-4}{2}$	⋯
3	$\frac{0}{3}$	$\frac{+1}{3}$	$\frac{-1}{3}$	$\frac{+2}{3}$	$\frac{-2}{3}$	$\frac{+3}{3}$	$\frac{-3}{3}$	$\frac{+4}{3}$	$\frac{-4}{3}$	⋯
4	$\frac{0}{4}$	$\frac{+1}{4}$	$\frac{-1}{4}$	$\frac{+2}{4}$	$\frac{-2}{4}$	$\frac{+3}{4}$	$\frac{-3}{4}$	$\frac{+4}{4}$	$\frac{-4}{4}$	⋯
5	$\frac{0}{5}$	$\frac{+1}{5}$	$\frac{-1}{5}$	$\frac{+2}{5}$	$\frac{-2}{5}$	$\frac{+3}{5}$	$\frac{-3}{5}$	$\frac{+4}{5}$	$\frac{-4}{5}$	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

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$\mathbb{Q}$	$\mathbb{Z}$: 0	+1	−1	+2	−2	+3	−3	+4	−4	⋯
1	$\frac{0}{1}$	$\frac{+1}{1}$	$\frac{-1}{1}$	$\frac{+2}{1}$	$\frac{-2}{1}$	$\frac{+3}{1}$	$\frac{-3}{1}$	$\frac{+4}{1}$	$\frac{-4}{1}$	⋯
2	$\frac{0}{2}$	$\frac{+1}{2}$	$\frac{-1}{2}$	$\frac{+2}{2}$	$\frac{-2}{2}$	$\frac{+3}{2}$	$\frac{-3}{2}$	$\frac{+4}{2}$	$\frac{-4}{2}$	⋯
$\mathbb{N}$ 3	$\frac{0}{3}$	$\frac{+1}{3}$	$\frac{-1}{3}$	$\frac{+2}{3}$	$\frac{-2}{3}$	$\frac{+3}{3}$	$\frac{-3}{3}$	$\frac{+4}{3}$	$\frac{-4}{3}$	⋯
4	$\frac{0}{4}$	$\frac{+1}{4}$	$\frac{-1}{4}$	$\frac{+2}{4}$	$\frac{-2}{4}$	$\frac{+3}{4}$	$\frac{-3}{4}$	$\frac{+4}{4}$	$\frac{-4}{4}$	⋯
5	$\frac{0}{5}$	$\frac{+1}{5}$	$\frac{-1}{5}$	$\frac{+2}{5}$	$\frac{-2}{5}$	$\frac{+3}{5}$	$\frac{-3}{5}$	$\frac{+4}{5}$	$\frac{-4}{5}$	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

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$\mathbb{Q}$	$\mathbb{Z}$: 0	$+1$	-1	$+2$	-2	$+3$	-3	$+4$	-4	$\cdots$
1	$\frac{0}{1}$	$\frac{+1}{1}$	$\frac{-1}{1}$	$\frac{+2}{1}$	$\frac{-2}{1}$	$\frac{+3}{1}$	$\frac{-3}{1}$	$\frac{+4}{1}$	$\frac{-4}{1}$	$\cdots$
2	$\frac{0}{2}$	$\frac{+1}{2}$	$\frac{-1}{2}$	$\frac{+2}{2}$	$\frac{-2}{2}$	$\frac{+3}{2}$	$\frac{-3}{2}$	$\frac{+4}{2}$	$\frac{-4}{2}$	$\cdots$
$\mathbb{N}$ 3	$\frac{0}{3}$	$\frac{+1}{3}$	$\frac{-1}{3}$	$\frac{+2}{3}$	$\frac{-2}{3}$	$\frac{+3}{3}$	$\frac{-3}{3}$	$\frac{+4}{3}$	$\frac{-4}{3}$	$\cdots$
4	$\frac{0}{4}$	$\frac{+1}{4}$	$\frac{-1}{4}$	$\frac{+2}{4}$	$\frac{-2}{4}$	$\frac{+3}{4}$	$\frac{-3}{4}$	$\frac{+4}{4}$	$\frac{-4}{4}$	$\cdots$
5	$\frac{0}{5}$	$\frac{+1}{5}$	$\frac{-1}{5}$	$\frac{+2}{5}$	$\frac{-2}{5}$	$\frac{+3}{5}$	$\frac{-3}{5}$	$\frac{+4}{5}$	$\frac{-4}{5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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$|\,\mathbb{Q}\,| = |\,\mathbb{N}\,| \times |\,\mathbb{Z}\,| \gg |\,\mathbb{N}\,|$. ✘

$$\mathbb{Q} = \{\tfrac{0}{1}, \tfrac{+1}{1}, \tfrac{+1}{2}, \tfrac{0}{2}, \tfrac{0}{3}, \tfrac{+1}{3}, \tfrac{-1}{3}, \tfrac{-1}{2}, \tfrac{-1}{1}, \tfrac{+2}{1}, \tfrac{+2}{2}, \tfrac{+2}{3}, \tfrac{+2}{4}, \tfrac{-1}{4}, \tfrac{+1}{4}, \tfrac{0}{4}, \tfrac{0}{5}, \ldots\}$$

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$\mathbb{Q}$	$\mathbb{Z}$: 0	$+1$	-1	$+2$	-2	$+3$	-3	$+4$	-4	$\cdots$
1	$\frac{0}{1}$	$\frac{+1}{1}$	$\frac{-1}{1}$	$\frac{+2}{1}$	$\frac{-2}{1}$	$\frac{+3}{1}$	$\frac{-3}{1}$	$\frac{+4}{1}$	$\frac{-4}{1}$	$\cdots$
2	$\frac{0}{2}$	$\frac{+1}{2}$	$\frac{-1}{2}$	$\frac{+2}{2}$	$\frac{-2}{2}$	$\frac{+3}{2}$	$\frac{-3}{2}$	$\frac{+4}{2}$	$\frac{-4}{2}$	$\cdots$
$\mathbb{N}$ 3	$\frac{0}{3}$	$\frac{+1}{3}$	$\frac{-1}{3}$	$\frac{+2}{3}$	$\frac{-2}{3}$	$\frac{+3}{3}$	$\frac{-3}{3}$	$\frac{+4}{3}$	$\frac{-4}{3}$	$\cdots$
4	$\frac{0}{4}$	$\frac{+1}{4}$	$\frac{-1}{4}$	$\frac{+2}{4}$	$\frac{-2}{4}$	$\frac{+3}{4}$	$\frac{-3}{4}$	$\frac{+4}{4}$	$\frac{-4}{4}$	$\cdots$
5	$\frac{0}{5}$	$\frac{+1}{5}$	$\frac{-1}{5}$	$\frac{+2}{5}$	$\frac{-2}{5}$	$\frac{+3}{5}$	$\frac{-3}{5}$	$\frac{+4}{5}$	$\frac{-4}{5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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$|\mathbb{Q}| = |\mathbb{N}| \times |\mathbb{Z}| \gg |\mathbb{N}|$. ✘

$$\mathbb{Q} = \{\tfrac{0}{1}, \tfrac{+1}{1}, \tfrac{+1}{2}, \tfrac{0}{2}, \tfrac{0}{3}, \tfrac{+1}{3}, \tfrac{-1}{3}, \tfrac{-1}{2}, \tfrac{-1}{1}, \tfrac{+2}{1}, \tfrac{+2}{2}, \tfrac{+2}{3}, \tfrac{+2}{4}, \tfrac{-1}{4}, \tfrac{+1}{4}, \tfrac{0}{4}, \tfrac{0}{5}, \ldots\}$$

$|\{\text{Rational Values}\}| \le |\mathbb{Q}| \le |\mathbb{N}|.$

Exercise. What is a mathematical formula for the list-position of $z/n \in \mathbb{Q}$?

Programs are Countable

Programs are finite binary strings. We show that all finite binary strings $\mathcal{B}$ are countable.

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Pop Quiz. What is the list-position of 0110?

Exercise. For the $(k+1)$-bit string $b = b_k b_{k-1} \cdots b_1 b_0$, define the strings numerical value:

$$\text{value}(b) = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \cdots + b_{k-1} \cdot 2^{k-1} + b_k \cdot 2^k.$$

Show:

$$\text{list-position of } b \ = \ 2^{\text{length}(b)} + \text{value}(b).$$

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Show:

$$\text{list-position of } b \;=\; 2^{\text{length}(b)} + \text{value}(b).$$

$\mathbb{N}_0, E, \mathbb{Z}, \mathbb{Q}, \mathcal{B}$ are countable,... **Is Everything Countable?**

Infinite Binary Strings are Uncountable

Cantor's Diagonal Argument: Assume there is a list of *all* infinite binary strings.

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b_1:	**0**	0	0	1	0	0	0	0	0	0	0	0	0	0	0···
b_2:	0	**0**	1	1	0	1	0	0	1	0	0	0	0	1	0···
b_3:	1	1	**0**	0	0	0	1	0	0	0	0	1	1	0	0···
b_4:	1	0	1	**0**	0	1	0	0	0	0	1	0	0	0	0···
b_5:	0	1	1	0	**1**	0	1	0	0	0	0	0	0	0	0···
b_6:	0	1	0	1	1	**0**	0	0	0	1	0	0	0	0	0···
b_7:	0	0	1	0	0	1	**0**	0	0	0	0	0	0	0	1···
b_8:	0	0	1	0	1	1	0	**1**	0	0	0	0	1	0	0···
b_9:	0	0	0	0	1	1	0	0	**0**	0	1	0	0	0	0···
b_{10}:	1	0	1	1	1	0	1	0	0	**1**	1	0	0	0	0···
⋮															

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b_3:	1	1	**0**	0	0	0	1	0	0	0	0	1	1	0	0$\cdots$
b_4:	1	0	1	**0**	0	1	0	0	0	0	1	0	0	0	0$\cdots$
b_5:	0	1	1	0	**1**	0	1	0	0	0	0	0	0	0	0$\cdots$
b_6:	0	1	0	1	1	**0**	0	0	0	1	0	0	0	0	0$\cdots$
b_7:	0	0	1	0	0	1	**0**	0	0	0	0	0	0	0	1$\cdots$
b_8:	0	0	1	0	1	1	0	**1**	0	0	0	0	1	0	0$\cdots$
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$\vdots$															

Consider the "diagonal string"

$$b = 0000100101\cdots$$

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$$
\begin{array}{r@{\;}ccccccccccccccc}
b_1: & \mathbf{0} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots \\
b_2: & 0 & \mathbf{0} & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0\cdots \\
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\vdots & & & & & & & & & & & & & & &
\end{array}
$$

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$\bar{b}$ is not in the list (differs in the ith position from b_i), a contradiction.

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$$
\begin{array}{r@{\;}l}
b_1: & \mathbf{0}\;0\;0\;1\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\cdots \\
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Reals are Uncountable

Every real has an infinite binary representation and every infinite binary string evaluates to a real number.

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Every real has an infinite binary representation and every infinite binary string evaluates to a real number.

$$\text{e.g.} \qquad 0.0011111111111111\cdots = \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \cdots = \frac{1}{2}.$$

That is $|\{\text{reals in } [0,1]\}| = |\{\text{infinte binary stings}\}| > |\,\mathbb{N}\,|$.

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Every program is a finite binary string. For example,

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int main();     //a program that does nothing
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is the finite binary string (ASCII code)

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Functions $\leftarrow$ Uncountable $\quad\rightarrow\quad |\{\text{functions on } \mathbb{N}\}| \gg |\{\text{programs}\}|$

There are MANY MANY functions that cannot be computed by programs!
Are there interesting, useful functions that cannot be computed by programs?