

Foundations of Computer Science

Lecture 19

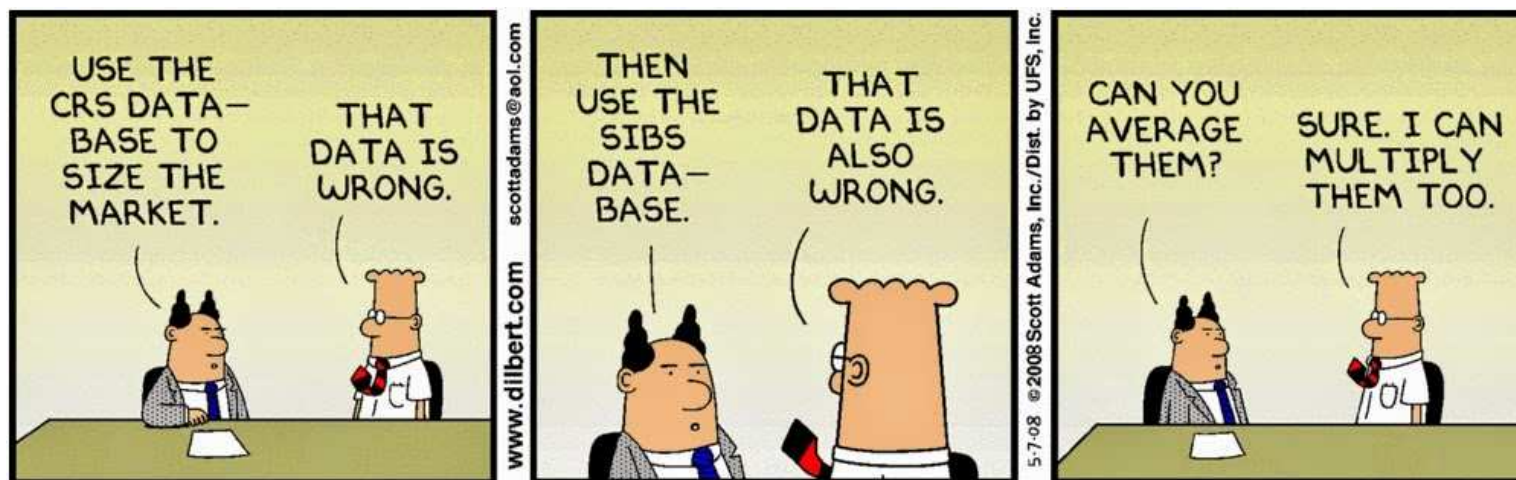
Expected Value

The Average Over Many Runs of an Experiment

Mathematical Expectation: A Number that Summarizes a PDF

Conditional Expectation

Law of Total Expectation



① Random variables.

- ▶ PDF.
- ▶ CDF.
- ▶ Joint-PDF.
- ▶ Independent random variables.

② Important random variables.

- ▶ Bernoulli (indicator).
- ▶ Uniform (equalizer in strategic games).
- ▶ Binomial (sum of Bernoullis, e.g. number of heads in n coin tosses).
- ▶ Exponential Waiting Time Distribution (repeated tries till success).

Today: Expected Value

1 Expected value approximates the sample average.

2 Mathematical Expectation

3 Examples

- Sum of dice.
- Bernoulli.
- Uniform.
- Binomial.
- Waiting time.

4 Conditional Expectation

5 Law of Total Expectation

Sample Average: Toss Two Coins Many Times

ω	SAMPLE SPACE Ω					$\rightarrow$	x	$x \in \mathbf{X}(\Omega)$		
	HH	HT	TH	TT				0	1	2
$P(\omega)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$				$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$\mathbf{X}(\omega)$	2	1	1	0	$\leftarrow$ number of heads					

Sample Average: Toss Two Coins Many Times

SAMPLE SPACE Ω					$\rightarrow$	$x \in \mathbf{X}(\Omega)$			
ω	HH	HT	TH	TT		x	0	1	2
$P(\omega)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$		$P_{\mathbf{X}}(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$\mathbf{X}(\omega)$	2	1	1	0	$\leftarrow$ number of heads				

Toss two coins and repeat the experiment $n = 24$ times:

HH	TH	HT	HH	HH	TH	TT	TT	HH	TT	HT	HT	HH	HT	TT	HT	TT	HT	HT	TH	HH	TH	TT	TH
2	1	1	2	2	1	0	0	2	0	1	1	2	1	0	1	0	1	1	1	2	1	0	1

Sample Average: Toss Two Coins Many Times

ω	SAMPLE SPACE Ω				$\rightarrow$	x	$x \in \mathbf{X}(\Omega)$		
	HH	HT	TH	TT			0	1	2
$P(\omega)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$		$P_{\mathbf{X}}(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$\mathbf{X}(\omega)$	2	1	1	0	$\leftarrow$ number of heads				

Toss two coins and repeat the experiment $n = 24$ times:

HH	TH	HT	HH	HH	TH	TT	TT	HH	TT	HT	HT	HH	HT	TT	HT	TT	HT	HT	TH	HH	TH	TT	TH
2	1	1	2	2	1	0	0	2	0	1	1	2	1	0	1	0	1	1	1	2	1	0	1

Average value of $\mathbf{X}$:

$$\frac{2 + 1 + 1 + 2 + 2 + 1 + 0 + 0 + 2 + 0 + 1 + 1 + 2 + 1 + 0 + 1 + 0 + 1 + 1 + 1 + 2 + 1 + 0 + 1}{24} = \frac{24}{24} = 1.$$

Sample Average: Toss Two Coins Many Times

ω	SAMPLE SPACE Ω					$\rightarrow$	x	$x \in \mathbf{X}(\Omega)$		
	HH	HT	TH	TT				0	1	2
$P(\omega)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$			$P_{\mathbf{X}}(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$\mathbf{X}(\omega)$	2	1	1	0	$\leftarrow$ number of heads					

Toss two coins and repeat the experiment $n = 24$ times:

HH	TH	HT	HH	HH	TH	TT	TT	HH	TT	HT	HT	HH	HT	TT	HT	TT	HT	HT	TH	HH	TH	TT	TH
2	1	1	2	2	1	0	0	2	0	1	1	2	1	0	1	0	1	1	1	2	1	0	1

Average value of $\mathbf{X}$:

$$\frac{2 + 1 + 1 + 2 + 2 + 1 + 0 + 0 + 2 + 0 + 1 + 1 + 2 + 1 + 0 + 1 + 0 + 1 + 1 + 1 + 2 + 1 + 0 + 1}{24} = \frac{24}{24} = 1.$$

Re-order outcomes:

TT	TT	TT	TT	TT	TT	HT	HT	HT	HT	HT	HT	HT	TH	TH	TH	TH	TH	HH	HH	HH	HH	HH	HH
$n_0 = 6$						$n_1 = 12$						$n_2 = 6$											
0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2

Sample Average: Toss Two Coins Many Times

ω	SAMPLE SPACE Ω					$\rightarrow$	x	$x \in \mathbf{X}(\Omega)$		
	HH	HT	TH	TT				0	1	2
$P(\omega)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$				$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$\mathbf{X}(\omega)$	2	1	1	0	$\leftarrow$ number of heads					

Toss two coins and repeat the experiment $n = 24$ times:

HH	TH	HT	HH	HH	TH	TT	TT	HH	TT	HT	HT	HH	HT	TT	HT	TT	HT	HT	TH	HH	TH	TT	TH
2	1	1	2	2	1	0	0	2	0	1	1	2	1	0	1	0	1	1	1	2	1	0	1

Average value of $\mathbf{X}$:

$$\frac{2 + 1 + 1 + 2 + 2 + 1 + 0 + 0 + 2 + 0 + 1 + 1 + 2 + 1 + 0 + 1 + 0 + 1 + 1 + 1 + 2 + 1 + 0 + 1}{24} = \frac{24}{24} = 1.$$

Re-order outcomes:

TT	TT	TT	TT	TT	TT	HT	HT	HT	HT	HT	HT	HT	TH	TH	TH	TH	TH	HH	HH	HH	HH	HH	HH
$n_0 = 6$						$n_1 = 12$						$n_2 = 6$											
0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2

Average value of $\mathbf{X}$:

$$\frac{6 \times 0 + 12 \times 1 + 6 \times 2}{n} = \frac{24}{24}$$

Mathematical Expectation of a Random Variable **X**

TT	TT	TT	TT	TT	TT	HT	HT	HT	HT	HT	HT	HT	TH	TH	TH	TH	TH	HH	HH	HH	HH	HH	HH
$n_0 = 6$						$n_1 = 12$						$n_2 = 6$											
0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2

Mathematical Expectation of a Random Variable **X**

TT	TT	TT	TT	TT	TT	HT	HT	HT	HT	HT	HT	HT	TH	TH	TH	TH	TH	HH	HH	HH	HH	HH	HH
$n_0 = 6$						$n_1 = 12$						$n_2 = 6$											
0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2

Average value of **X**:

$$\frac{n_0 \times 0 + n_1 \times 1 + n_2 \times 2 + n_3 \times 3}{n}$$

Mathematical Expectation of a Random Variable **X**

TT	TT	TT	TT	TT	TT	HT	HT	HT	HT	HT	HT	HT	TH	TH	TH	TH	TH	HH	HH	HH	HH	HH	HH
$n_0 = 6$						$n_1 = 12$						$n_2 = 6$											
0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2

Average value of **X**:

$$\frac{n_0 \times 0 + n_1 \times 1 + n_2 \times 2 + n_3 \times 3}{n} = \frac{n_0}{n} \times 0 + \frac{n_1}{n} \times 1 + \frac{n_2}{n} \times 2$$

$\uparrow$
 $\approx P_X(0)$

$\uparrow$
 $\approx P_X(1)$

$\uparrow$
 $\approx P_X(2)$

Mathematical Expectation of a Random Variable **X**

TT	TT	TT	TT	TT	TT	HT	HT	HT	HT	HT	HT	HT	TH	TH	TH	TH	TH	HH	HH	HH	HH	HH	HH
$n_0 = 6$						$n_1 = 12$						$n_2 = 6$											
0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2

Average value of **X**:

$$\begin{aligned} \frac{n_0 \times 0 + n_1 \times 1 + n_2 \times 2 + n_3 \times 3}{n} &= \frac{n_0}{n} \times 0 + \frac{n_1}{n} \times 1 + \frac{n_2}{n} \times 2 \\ &\quad \begin{matrix} \uparrow \\ \approx P_X(0) \end{matrix} \quad \begin{matrix} \uparrow \\ \approx P_X(1) \end{matrix} \quad \begin{matrix} \uparrow \\ \approx P_X(2) \end{matrix} \\ &\approx P_X(0) \times 0 + P_X(1) \times 1 + P_X(2) \times 2 \end{aligned}$$

Mathematical Expectation of a Random Variable **X**

TT	TT	TT	TT	TT	TT	HT	HT	HT	HT	HT	HT	HT	TH	TH	TH	TH	TH	HH	HH	HH	HH	HH	HH
$n_0 = 6$						$n_1 = 12$						$n_2 = 6$											
0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2

Average value of **X**:

$$\begin{aligned} \frac{n_0 \times 0 + n_1 \times 1 + n_2 \times 2 + n_3 \times 3}{n} &= \frac{n_0}{n} \times 0 + \frac{n_1}{n} \times 1 + \frac{n_2}{n} \times 2 \\ &\quad \uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow \\ &\quad \approx P_X(0) \qquad \approx P_X(1) \qquad \approx P_X(2) \\ &\approx P_X(0) \times 0 + P_X(1) \times 1 + P_X(2) \times 2 \\ &= \sum_{x \in \mathbf{X}(\Omega)} x \cdot P_X(x) \end{aligned}$$

For two coins the expected value is $0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1$.

Add the possible values x weighted by their probabilities $P_X(x)$,

$$\mathbb{E}[\mathbf{X}] = \sum_{x \in \mathbf{X}(\Omega)} x \cdot P_X(x).$$

Synonyms: Expectation; Expected Value; Mean; Average.

Important Exercise. Show that $\mathbb{E}[\mathbf{X}] = \sum_{\omega \in \Omega} \mathbf{X}(\omega)P(\omega)$.

Expected Number of Heads from 3 Coin Tosses is $1\frac{1}{2}$!

$$\mathbb{E}[\mathbf{X}] = \sum_{x \in \mathbf{X}(\Omega)} x \cdot P_{\mathbf{X}}(x).$$

Expected Number of Heads from 3 Coin Tosses is $1\frac{1}{2}$!

$$\mathbb{E}[\mathbf{X}] = \sum_{x \in \mathbf{X}(\Omega)} x \cdot P_{\mathbf{X}}(x).$$

	SAMPLE SPACE Ω							
ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0

→

	$x \in \mathbf{X}(\Omega)$			
x	0	1	2	3
$P_{\mathbf{X}}(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Expected Number of Heads from 3 Coin Tosses is $1\frac{1}{2}$!

$$\mathbb{E}[\mathbf{X}] = \sum_{x \in \mathbf{X}(\Omega)} x \cdot P_{\mathbf{X}}(x).$$

	SAMPLE SPACE Ω								
ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT	
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0	

→

	$x \in \mathbf{X}(\Omega)$			
x	0	1	2	3
$P_{\mathbf{X}}(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\begin{aligned} \mathbb{E}[\text{number heads}] &= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} \\ &= \frac{12}{8} \\ &= 1\frac{1}{2}. \end{aligned}$$

What does this mean?!?

Exercise. Let $\mathbf{X}$ be the the value of a fair die roll. Show that $\mathbb{E}[\mathbf{X}] = 3\frac{1}{2}$.

Expected Sum of Two Dice

<u>Probability Space</u>							<u>X = sum</u>							
Die 2 Value		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		7	8	9	10	11	12
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		6	7	8	9	10	11
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		5	6	7	8	9	10
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		4	5	6	7	8	9
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		3	4	5	6	7	8
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		2	3	4	5	6	7
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$							
     							     							
Die 1 Value							Die 1 Value							

x	2	3	4	5	6	7	8	9	10	11	12
$P_{\mathbf{X}}(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Expected Sum of Two Dice

<u>Probability Space</u>							<u>X = sum</u>							
Die 2 Value		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		7	8	9	10	11	12
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		6	7	8	9	10	11
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		5	6	7	8	9	10
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		4	5	6	7	8	9
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		3	4	5	6	7	8
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		2	3	4	5	6	7
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$							
     							     							
Die 1 Value							Die 1 Value							

x	2	3	4	5	6	7	8	9	10	11	12
$P_{\mathbf{X}}(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\mathbb{E}[\mathbf{X}] = \sum_x x \cdot P_{\mathbf{X}}(x)$$

Expected Sum of Two Dice

Probability Space							<u>X = sum</u>							
Die 2 Value		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		7	8	9	10	11	12
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		6	7	8	9	10	11
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		5	6	7	8	9	10
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		4	5	6	7	8	9
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		3	4	5	6	7	8
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		2	3	4	5	6	7
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$							
     							     							
Die 1 Value							Die 1 Value							

x	2	3	4	5	6	7	8	9	10	11	12
$P_{\mathbf{X}}(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\begin{aligned} \mathbb{E}[\mathbf{X}] &= \sum_x x \cdot P_{\mathbf{X}}(x) \\ &= \frac{1}{36}(2 \times 1 + 3 \times 2 + 4 \times 3 + 5 \times 4 + 6 \times 5 + 7 \times 6 + 8 \times 5 + 9 \times 4 + 10 \times 3 + 11 \times 2 + 12 \times 1) \end{aligned}$$

Expected Sum of Two Dice

<u>Probability Space</u>							<u>X = sum</u>																		
Die 2 Value		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$		7	8	9	10	11	12											
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$											
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$											
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$											
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$											
		$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$											
																									
Die 1 Value							Die 1 Value																		

$P_{\mathbf{X}}(x)$	x	2	3	4	5	6	7	8	9	10	11	12
		$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

x	2	3	4	5	6	7	8	9	10	11	12
$P_{\mathbf{X}}(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\begin{aligned}
 \mathbb{E}[\mathbf{X}] &= \sum_x x \cdot P_{\mathbf{X}}(x) \\
 &= \frac{1}{36}(2 \times 1 + 3 \times 2 + 4 \times 3 + 5 \times 4 + 6 \times 5 + 7 \times 6 + 8 \times 5 + 9 \times 4 + 10 \times 3 + 11 \times 2 + 12 \times 1) \\
 &= \frac{252}{36} = 7.
 \end{aligned}$$

(Expected sum of two dice is twice the expected roll of one die.)

Expected Value of a Bernoulli Random Variable

A Bernoulli random variable $\mathbf{X}$ takes a value in $\{0, 1\}$

$$\begin{array}{c|cc} x & 0 & 1 \\ \hline P_{\mathbf{X}}(x) & 1-p & p \end{array}$$

Expected Value of a Bernoulli Random Variable

A Bernoulli random variable $\mathbf{X}$ takes a value in $\{0, 1\}$

$$\begin{array}{c|cc} x & 0 & 1 \\ \hline P_{\mathbf{X}}(x) & 1-p & p \end{array}$$

The expected value is

$$\mathbb{E}[\mathbf{X}] = 0 \cdot (1-p) + 1 \cdot p = p.$$

A Bernoulli random variable with success probability p has expected value p .

Expected Value of a Uniform Random Variable

A uniform random variable $\mathbf{X}$ takes values in $\{1, \dots, n\}$,

x	1	2	3	4	$\dots$	$n - 1$	n
$P_{\mathbf{X}}(x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$

Expected Value of a Uniform Random Variable

A uniform random variable $\mathbf{X}$ takes values in $\{1, \dots, n\}$,

$$\begin{array}{c|cccccc} x & 1 & 2 & 3 & 4 & \dots & n-1 & n \\ P_{\mathbf{X}}(x) & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \end{array}$$

The expected value is

$$\mathbb{E}[\mathbf{X}] = \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \dots + \frac{n}{n}$$

Expected Value of a Uniform Random Variable

A uniform random variable $\mathbf{X}$ takes values in $\{1, \dots, n\}$,

x	1	2	3	4	$\dots$	$n - 1$	n
$P_{\mathbf{X}}(x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$

The expected value is

$$\begin{aligned}\mathbb{E}[\mathbf{X}] &= \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \dots + \frac{n}{n} \\ &= \frac{1}{n}(1 + 2 + \dots + n)\end{aligned}$$

Expected Value of a Uniform Random Variable

A uniform random variable $\mathbf{X}$ takes values in $\{1, \dots, n\}$,

x	1	2	3	4	$\dots$	$n - 1$	n
$P_{\mathbf{X}}(x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$

The expected value is

$$\begin{aligned}\mathbb{E}[\mathbf{X}] &= \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \dots + \frac{n}{n} \\ &= \frac{1}{n}(1 + 2 + \dots + n) \\ &= \frac{1}{n} \times \frac{1}{2}n(n + 1).\end{aligned}$$

A uniform random variable on $[1, n]$ has expected value $= \frac{1}{2}(n + 1)$.

What is the Expected Number of Heads in n Coin Tosses?

Binomial distribution: $P_X(k) = B(k; n, p) = \binom{n}{k} p^k (1 - p)^{n-k}$.

x	0	1	2	...	k	...	n
$P_X(x)$	$\binom{n}{0} p^0 (1 - p)^n$	$\binom{n}{1} p^1 (1 - p)^{n-1}$	$\binom{n}{2} p^2 (1 - p)^{n-2}$	...	$\binom{n}{k} p^k (1 - p)^{n-k}$	...	$\binom{n}{n} p^n (1 - p)^0$

What is the Expected Number of Heads in n Coin Tosses?

Binomial distribution: $P_{\mathbf{X}}(k) = B(k; n, p) = \binom{n}{k} p^k (1 - p)^{n-k}$.

x	0	1	2	...	k	...	n
$P_{\mathbf{X}}(x)$	$\binom{n}{0} p^0 (1 - p)^n$	$\binom{n}{1} p^1 (1 - p)^{n-1}$	$\binom{n}{2} p^2 (1 - p)^{n-2}$	...	$\binom{n}{k} p^k (1 - p)^{n-k}$	...	$\binom{n}{n} p^n (1 - p)^0$

$$\mathbb{E}[\mathbf{X}] = 0 \cdot \binom{n}{0} p^0 q^n + 1 \cdot \binom{n}{1} p^1 q^{n-1} + \dots + k \cdot \binom{n}{k} p^k q^{n-k} + \dots + n \cdot \binom{n}{n} p^n q^0 \qquad (q = 1 - p)$$

What is the Expected Number of Heads in n Coin Tosses?

Binomial distribution: $P_{\mathbf{X}}(k) = B(k; n, p) = \binom{n}{k} p^k (1 - p)^{n-k}$.

x	0	1	2	...	k	...	n
$P_{\mathbf{X}}(x)$	$\binom{n}{0} p^0 (1 - p)^n$	$\binom{n}{1} p^1 (1 - p)^{n-1}$	$\binom{n}{2} p^2 (1 - p)^{n-2}$	...	$\binom{n}{k} p^k (1 - p)^{n-k}$	...	$\binom{n}{n} p^n (1 - p)^0$

$$\mathbb{E}[\mathbf{X}] = 0 \cdot \binom{n}{0} p^0 q^n + 1 \cdot \binom{n}{1} p^1 q^{n-1} + \dots + k \cdot \binom{n}{k} p^k q^{n-k} + \dots + n \cdot \binom{n}{n} p^n q^0 \qquad (q = 1 - p)$$

Binomial Theorem:

$$(p + q)^n = \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \dots + \binom{n}{k} p^k q^{n-k} + \dots + \binom{n}{n} p^n q^0$$

What is the Expected Number of Heads in n Coin Tosses?

Binomial distribution: $P_{\mathbf{X}}(k) = B(k; n, p) = \binom{n}{k} p^k (1 - p)^{n-k}$.

$$\begin{array}{c|cccccccc}
 x & 0 & 1 & 2 & \dots & k & \dots & n \\
 P_{\mathbf{X}}(x) & \binom{n}{0} p^0 (1-p)^n & \binom{n}{1} p^1 (1-p)^{n-1} & \binom{n}{2} p^2 (1-p)^{n-2} & \dots & \binom{n}{k} p^k (1-p)^{n-k} & \dots & \binom{n}{n} p^n (1-p)^0
 \end{array}$$

$$\mathbb{E}[\mathbf{X}] = 0 \cdot \binom{n}{0} p^0 q^n + 1 \cdot \binom{n}{1} p^1 q^{n-1} + \dots + k \cdot \binom{n}{k} p^k q^{n-k} + \dots + n \cdot \binom{n}{n} p^n q^0 \qquad (q = 1 - p)$$

Binomial Theorem:

$$\begin{aligned}
 (p + q)^n &= \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \dots + \binom{n}{k} p^k q^{n-k} + \dots + \binom{n}{n} p^n q^0 \\
 \xrightarrow{\frac{d}{dp}} \quad n(p + q)^{n-1} &= 1 \cdot \binom{n}{1} p^0 q^{n-1} + 2 \cdot \binom{n}{2} p^1 q^{n-2} + \dots + k \cdot \binom{n}{k} p^{k-1} q^{n-k} + \dots + n \cdot \binom{n}{n} p^{n-1} q^0
 \end{aligned}$$

What is the Expected Number of Heads in n Coin Tosses?

Binomial distribution: $P_{\mathbf{X}}(k) = B(k; n, p) = \binom{n}{k} p^k (1 - p)^{n-k}$.

x	0	1	2	...	k	...	n
$P_{\mathbf{X}}(x)$	$\binom{n}{0} p^0 (1 - p)^n$	$\binom{n}{1} p^1 (1 - p)^{n-1}$	$\binom{n}{2} p^2 (1 - p)^{n-2}$	...	$\binom{n}{k} p^k (1 - p)^{n-k}$	...	$\binom{n}{n} p^n (1 - p)^0$

$$\mathbb{E}[\mathbf{X}] = 0 \cdot \binom{n}{0} p^0 q^n + 1 \cdot \binom{n}{1} p^1 q^{n-1} + \dots + k \cdot \binom{n}{k} p^k q^{n-k} + \dots + n \cdot \binom{n}{n} p^n q^0 \qquad (q = 1 - p)$$

Binomial Theorem:

$$\begin{aligned} (p + q)^n &= \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \dots + \binom{n}{k} p^k q^{n-k} + \dots + \binom{n}{n} p^n q^0 \\ \xrightarrow{\frac{d}{dp}} \quad n(p + q)^{n-1} &= 1 \cdot \binom{n}{1} p^0 q^{n-1} + 2 \cdot \binom{n}{2} p^1 q^{n-2} + \dots + k \cdot \binom{n}{k} p^{k-1} q^{n-k} + \dots + n \cdot \binom{n}{n} p^{n-1} q^0 \\ \xrightarrow{\times p} \quad np \underbrace{(p + q)^{n-1}}_1 &= 1 \cdot \binom{n}{1} p^1 q^{n-1} + 2 \cdot \binom{n}{2} p^2 q^{n-2} + \dots + k \cdot \binom{n}{k} p^k q^{n-k} + \dots + n \cdot \binom{n}{n} p^n q^0 \end{aligned}$$

What is the Expected Number of Heads in n Coin Tosses?

Binomial distribution: $P_X(k) = B(k; n, p) = \binom{n}{k} p^k (1 - p)^{n-k}$.

x	0	1	2	...	k	...	n
$P_X(x)$	$\binom{n}{0} p^0 (1 - p)^n$	$\binom{n}{1} p^1 (1 - p)^{n-1}$	$\binom{n}{2} p^2 (1 - p)^{n-2}$	...	$\binom{n}{k} p^k (1 - p)^{n-k}$	...	$\binom{n}{n} p^n (1 - p)^0$

$$\mathbb{E}[\mathbf{X}] = 0 \cdot \binom{n}{0} p^0 q^n + 1 \cdot \binom{n}{1} p^1 q^{n-1} + \dots + k \cdot \binom{n}{k} p^k q^{n-k} + \dots + n \cdot \binom{n}{n} p^n q^0 \qquad (q = 1 - p)$$

Binomial Theorem:

$$\begin{aligned} (p + q)^n &= \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \dots + \binom{n}{k} p^k q^{n-k} + \dots + \binom{n}{n} p^n q^0 \\ \xrightarrow{\frac{d}{dp}} \quad n(p + q)^{n-1} &= 1 \cdot \binom{n}{1} p^0 q^{n-1} + 2 \cdot \binom{n}{2} p^1 q^{n-2} + \dots + k \cdot \binom{n}{k} p^{k-1} q^{n-k} + \dots + n \cdot \binom{n}{n} p^{n-1} q^0 \\ \xrightarrow{\times p} \quad np \underbrace{(p + q)^{n-1}}_1 &= 1 \cdot \binom{n}{1} p^1 q^{n-1} + 2 \cdot \binom{n}{2} p^2 q^{n-2} + \dots + k \cdot \binom{n}{k} p^k q^{n-k} + \dots + n \cdot \binom{n}{n} p^n q^0 \end{aligned}$$

Expected number of heads in n biased coin tosses is np .

Example. Answer randomly 15 multiple choice questions with 5 choices ($p = \frac{1}{5}$): expect to get $15 \times \frac{1}{5} = 3$ correct.

Expected Waiting Time to Success

Exponential Waiting Time Distribution: $P_X(t) = \beta(1 - p)^t$.

t	1	2	3	...	k	...	
$P_X(t)$	$\beta(1 - p)$	$\beta(1 - p)^2$	$\beta(1 - p)^3$	...	$\beta(1 - p)^k$	...	$(\beta = p/(1 - p))$

Expected Waiting Time to Success

Exponential Waiting Time Distribution: $P_{\mathbf{X}}(t) = \beta(1 - p)^t$.

$$P_{\mathbf{X}}(t) \quad \left| \quad \begin{array}{cccccc} 1 & 2 & 3 & \dots & k & \dots \\ \beta(1 - p) & \beta(1 - p)^2 & \beta(1 - p)^3 & \dots & \beta(1 - p)^k & \dots \end{array} \right. \quad (\beta = p/(1 - p))$$

$$\mathbb{E}[\mathbf{X}] = \beta(1 \cdot (1 - p)^1 + 2 \cdot (1 - p)^2 + 3 \cdot (1 - p)^3 + \dots + k \cdot (1 - p)^k + \dots)$$

Expected Waiting Time to Success

Exponential Waiting Time Distribution: $P_X(t) = \beta(1 - p)^t$.

t	1	2	3	...	k	...	
$P_X(t)$	$\beta(1 - p)$	$\beta(1 - p)^2$	$\beta(1 - p)^3$	...	$\beta(1 - p)^k$	...	$(\beta = p/(1 - p))$

$$\mathbb{E}[\mathbf{X}] = \beta(1 \cdot (1 - p)^1 + 2 \cdot (1 - p)^2 + 3 \cdot (1 - p)^3 + \dots + k \cdot (1 - p)^k + \dots)$$

Geometric series formula:

$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + a^4 + \dots$$

Expected Waiting Time to Success

Exponential Waiting Time Distribution: $P_X(t) = \beta(1 - p)^t$.

t	1	2	3	...	k	...	
$P_X(t)$	$\beta(1 - p)$	$\beta(1 - p)^2$	$\beta(1 - p)^3$	...	$\beta(1 - p)^k$	...	$(\beta = p/(1 - p))$

$$\mathbb{E}[\mathbf{X}] = \beta(1 \cdot (1 - p)^1 + 2 \cdot (1 - p)^2 + 3 \cdot (1 - p)^3 + \dots + k \cdot (1 - p)^k + \dots)$$

Geometric series formula:

$$\begin{aligned} \frac{1}{1-a} &= 1 + a + a^2 + a^3 + a^4 + \dots \\ \xrightarrow{\frac{d}{da}} \frac{1}{(1-a)^2} &= 1 + 2 \cdot a + 3 \cdot a^2 + 4 \cdot a^3 + 5a \cdot a^4 + \dots \end{aligned}$$

Expected Waiting Time to Success

Exponential Waiting Time Distribution: $P_X(t) = \beta(1 - p)^t$.

t	1	2	3	...	k	...	
$P_X(t)$	$\beta(1 - p)$	$\beta(1 - p)^2$	$\beta(1 - p)^3$	...	$\beta(1 - p)^k$	...	$(\beta = p/(1 - p))$

$$\mathbb{E}[\mathbf{X}] = \beta(1 \cdot (1 - p)^1 + 2 \cdot (1 - p)^2 + 3 \cdot (1 - p)^3 + \dots + k \cdot (1 - p)^k + \dots)$$

Geometric series formula:

	$\frac{1}{1-a}$	=	$1 + a + a^2 + a^3 + a^4 + \dots$	
$\frac{d}{da} \rightarrow$	$\frac{1}{(1-a)^2}$	=	$1 + 2 \cdot a + 3 \cdot a^2 + 4 \cdot a^3 + 5a \cdot a^4 + \dots$	
$\times a \rightarrow$	$\frac{a}{(1-a)^2}$	=	$1 \cdot a^1 + 2 \cdot a^2 + 3 \cdot a^3 + 4 \cdot a^4 + 5a \cdot a^5 + \dots$	$(a = 1 - p)$

Expected Waiting Time to Success

Exponential Waiting Time Distribution: $P_X(t) = \beta(1 - p)^t$.

$$P_X(t) \left| \begin{array}{cccccc} t & 1 & 2 & 3 & \dots & k & \dots \\ \beta(1 - p) & \beta(1 - p)^2 & \beta(1 - p)^3 & \dots & \beta(1 - p)^k & \dots \end{array} \right. \quad (\beta = p/(1 - p))$$

$$\mathbb{E}[X] = \beta(1 \cdot (1 - p)^1 + 2 \cdot (1 - p)^2 + 3 \cdot (1 - p)^3 + \dots + k \cdot (1 - p)^k + \dots)$$

Geometric series formula:

$$\begin{aligned} \frac{1}{1-a} &= 1 + a + a^2 + a^3 + a^4 + \dots \\ \xrightarrow{\frac{d}{da}} \frac{1}{(1-a)^2} &= 1 + 2 \cdot a + 3 \cdot a^2 + 4 \cdot a^3 + 5a \cdot a^4 + \dots \\ \xrightarrow{\times a} \frac{a}{(1-a)^2} &= 1 \cdot a^1 + 2 \cdot a^2 + 3 \cdot a^3 + 4 \cdot a^4 + 5a \cdot a^5 + \dots \quad (a = 1 - p) \\ \rightarrow \mathbb{E}[X] &= \beta \times \frac{1-p}{p^2} = \frac{1}{p}. \end{aligned}$$

Expected waiting time is $1/p$.

Exercise. A couple who is waiting for a boy expects to make 2 trials (children).

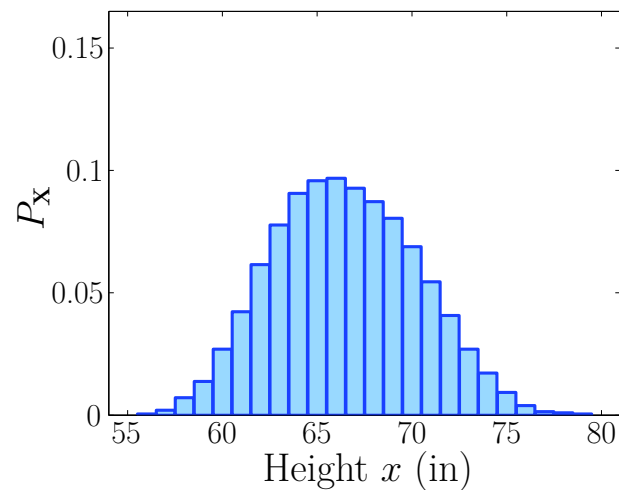
Conditional Expectation: Expected Height of Men

New information changes a probability. Hence, the expected value also changes.

Conditional Expectation: Expected Height of Men

New information changes a probability. Hence, the expected value also changes.

Height Distribution

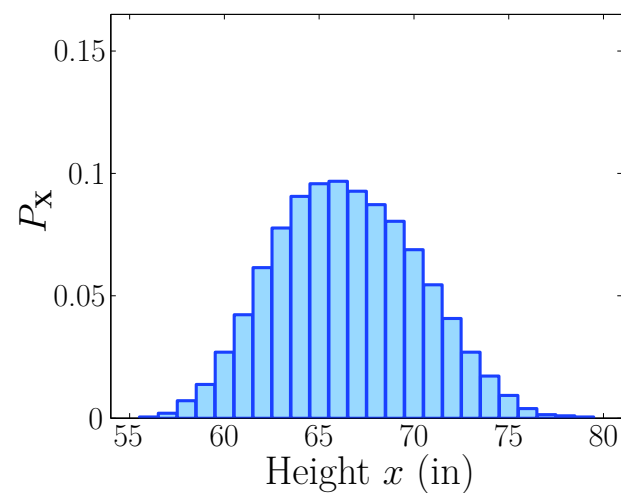


$$\mathbb{E}[\text{height}] \approx 66\frac{1}{2}''.$$

Conditional Expectation: Expected Height of Men

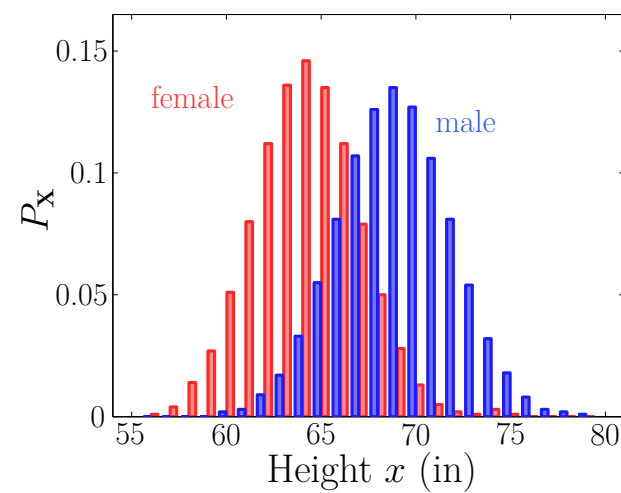
New information changes a probability. Hence, the expected value also changes.

Height Distribution



$$\mathbb{E}[\text{height}] \approx 66\frac{1}{2}''.$$

Conditional Height Distribution



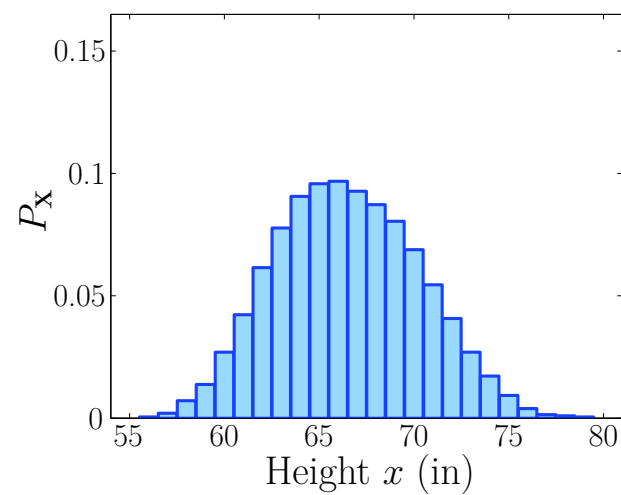
$$\mathbb{E}[\text{height} \mid \text{female}] \approx 64'' \text{ (red)}$$

$$\mathbb{E}[\text{height} \mid \text{male}] \approx 69'' \text{ (blue)}$$

Conditional Expectation: Expected Height of Men

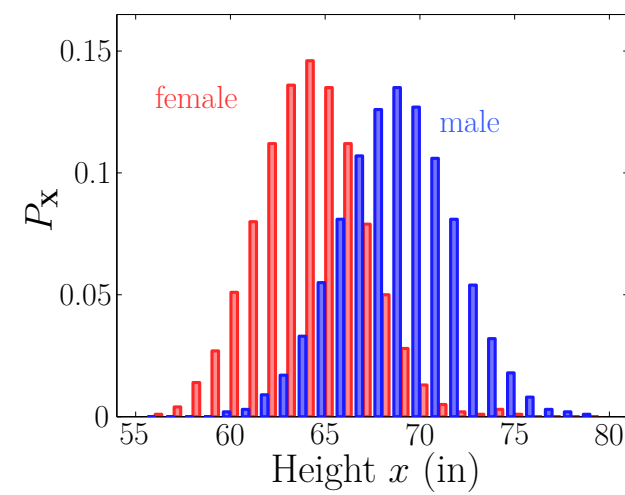
New information changes a probability. Hence, the expected value also changes.

Height Distribution



$$\mathbb{E}[\text{height}] \approx 66\frac{1}{2}''.$$

Conditional Height Distribution



$$\begin{aligned} \mathbb{E}[\text{height} \mid \text{female}] &\approx 64'' \text{ (red)} \\ \mathbb{E}[\text{height} \mid \text{male}] &\approx 69'' \text{ (blue)} \end{aligned}$$

Conditional Expected Value $\mathbb{E}[\mathbf{X} \mid A]$:

$$\mathbb{E}[\mathbf{X} \mid A] = \sum_{x \in \mathbf{X}(\Omega)} x \cdot \mathbb{P}[\mathbf{X} = x \mid A] = \sum_{x \in \mathbf{X}(\Omega)} x \cdot P_{\mathbf{X}}(x \mid A).$$

Law of Total Expectation

Case by case analysis for expectation (similar to the Law of Total Probability).

Law of Total Expectation

$$\mathbb{E}[\mathbf{X}] = \mathbb{E}[\mathbf{X} \mid A] \cdot \mathbb{P}[A] + \mathbb{E}[\mathbf{X} \mid \overline{A}] \cdot \mathbb{P}[\overline{A}].$$

Law of Total Expectation

Case by case analysis for expectation (similar to the Law of Total Probability).

Law of Total Expectation

$$\mathbb{E}[\mathbf{X}] = \mathbb{E}[\mathbf{X} \mid A] \cdot \mathbb{P}[A] + \mathbb{E}[\mathbf{X} \mid \overline{A}] \cdot \mathbb{P}[\overline{A}].$$

$$\begin{array}{ccccccc} \mathbb{E}[\text{height}] & = & \mathbb{E}[\text{height} \mid \text{male}] & \mathbb{P}[\text{male}] & + & \mathbb{E}[\text{height} \mid \text{female}] & \mathbb{P}[\text{female}] \\ & & \uparrow & \uparrow & & \uparrow & \uparrow \\ & & 69" & 0.49 & & 64" & 0.51 \end{array}$$

Law of Total Expectation

Case by case analysis for expectation (similar to the Law of Total Probability).

Law of Total Expectation

$$\mathbb{E}[\mathbf{X}] = \mathbb{E}[\mathbf{X} \mid A] \cdot \mathbb{P}[A] + \mathbb{E}[\mathbf{X} \mid \overline{A}] \cdot \mathbb{P}[\overline{A}].$$

$$\begin{aligned} \mathbb{E}[\text{height}] &= \underset{\substack{\uparrow \\ 69''}}{\mathbb{E}[\text{height} \mid \text{male}]} \underset{\substack{\uparrow \\ 0.49}}{\mathbb{P}[\text{male}]} + \underset{\substack{\uparrow \\ 64''}}{\mathbb{E}[\text{height} \mid \text{female}]} \underset{\substack{\uparrow \\ 0.51}}{\mathbb{P}[\text{female}]} \\ &= 69 \times 0.49 + 64 \times 0.51 \\ &\approx 66\frac{1}{2}'' . \end{aligned}$$

Example

A jar has 9 fair coins and 1 two-headed coin.

Example

A jar has 9 fair coins and 1 two-headed coin.

Choose a random coin and flip it 10 times.

X is the number of heads.

Example

A jar has 9 fair coins and 1 two-headed coin.

Choose a random coin and flip it 10 times.

$\mathbf{X}$ is the number of heads.

$$\begin{array}{ccccccc} \mathbb{E}[\mathbf{X}] & = & \mathbb{E}[\mathbf{X} \mid \text{fair}] & \mathbb{P}[\text{fair}] & + & \mathbb{E}[\mathbf{X} \mid \text{2-headed}] & \mathbb{P}[\text{2-headed}] \\ & & \uparrow & \uparrow & & \uparrow & \uparrow \\ & & 10 \times \frac{1}{2} & \frac{9}{10} & & 10 & \frac{1}{10} \end{array}$$

Example

A jar has 9 fair coins and 1 two-headed coin.

Choose a random coin and flip it 10 times.

$\mathbf{X}$ is the number of heads.

$$\begin{aligned}\mathbb{E}[\mathbf{X}] &= \mathbb{E}[\mathbf{X} \mid \text{fair}] \mathbb{P}[\text{fair}] + \mathbb{E}[\mathbf{X} \mid \text{2-headed}] \mathbb{P}[\text{2-headed}] \\ &\quad \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ 10 \times \frac{1}{2} & \frac{9}{10} & 10 & \frac{1}{10} \end{array} \\ &= 5 \times \frac{9}{10} + 10 \times \frac{1}{10} \\ &= 5\frac{1}{2}.\end{aligned}$$

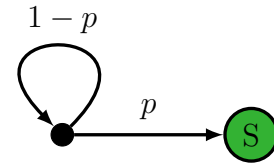
Expected Waiting Time from Law of Total Expectation

$\mathbf{X}$ is the waiting time.

Expected Waiting Time from Law of Total Expectation

$\mathbf{X}$ is the waiting time. Two cases:

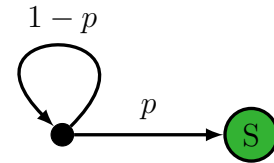
- First trial is a succeeds (S) with probability p , i.e., $\mathbf{X} = 1$.



Expected Waiting Time from Law of Total Expectation

$\mathbf{X}$ is the waiting time. Two cases:

- First trial is a succeeds (S) with probability p , i.e., $\mathbf{X} = 1$.
- First trial is a fails (F) with probability $1 - p$, i.e., “restart”.

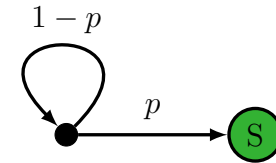


Expected Waiting Time from Law of Total Expectation

$\mathbf{X}$ is the waiting time. Two cases:

- First trial is a succeeds (S) with probability p , i.e., $\mathbf{X} = 1$.
- First trial is a fails (F) with probability $1 - p$, i.e., “restart”.

$$\begin{array}{ccccccc} \mathbb{E}[\mathbf{X}] & = & \mathbb{E}[\mathbf{X} \mid S] & \mathbb{P}[S] & + & \mathbb{E}[\mathbf{X} \mid F] & \mathbb{P}[F] \\ & & \uparrow & \uparrow & & \uparrow & \uparrow \\ & & 1 & p & & 1 + \mathbb{E}[\mathbf{X}] & (1-p) \end{array}$$



Expected Waiting Time from Law of Total Expectation

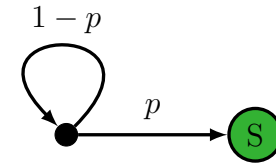
$\mathbf{X}$ is the waiting time. Two cases:

- First trial is a succeeds (S) with probability p , i.e., $\mathbf{X} = 1$.
- First trial is a fails (F) with probability $1 - p$, i.e., “restart”.

$$\begin{array}{ccccccc} \mathbb{E}[\mathbf{X}] & = & \mathbb{E}[\mathbf{X} \mid S] & \mathbb{P}[S] & + & \mathbb{E}[\mathbf{X} \mid F] & \mathbb{P}[F] \\ & & \uparrow & \uparrow & & \uparrow & \uparrow \\ & & 1 & p & & 1 + \mathbb{E}[\mathbf{X}] & (1-p) \end{array}$$

$$= p \cdot 1 + (1 - p) \cdot (1 + \mathbb{E}[\mathbf{X}])$$

$$= 1 + (1 - p) \cdot \mathbb{E}[\mathbf{X}].$$



Expected Waiting Time from Law of Total Expectation

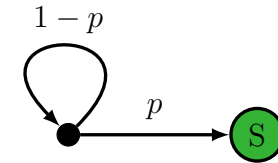
$\mathbf{X}$ is the waiting time. Two cases:

- First trial is a succeeds (S) with probability p , i.e., $\mathbf{X} = 1$.
- First trial is a fails (F) with probability $1 - p$, i.e., “restart”.

$$\begin{array}{ccccccc} \mathbb{E}[\mathbf{X}] & = & \mathbb{E}[\mathbf{X} \mid S] & \mathbb{P}[S] & + & \mathbb{E}[\mathbf{X} \mid F] & \mathbb{P}[F] \\ & & \uparrow & \uparrow & & \uparrow & \uparrow \\ & & 1 & p & & 1 + \mathbb{E}[\mathbf{X}] & (1 - p) \end{array}$$

$$= p \cdot 1 + (1 - p) \cdot (1 + \mathbb{E}[\mathbf{X}])$$

$$= 1 + (1 - p) \cdot \mathbb{E}[\mathbf{X}].$$



Solve for $\mathbb{E}[\mathbf{X}]$,

$$\mathbb{E}[\mathbf{X}] = \frac{1}{p}.$$

Practice. Exercise 6.5 and 6.8.