Foundations of Computer Science
Lecture 18

Random Variables

Measurable Outcomes
Probability Distribution Function
Bernoulli, Uniform, Binomial and Exponential Random Variables

Last Time

1. Independence.
    - Using independence to estimate complex probabilities.
2. Coincidence.
    - FOCS-twins.
    - The birthday paradox.
    - Application to hashing.
3. Random walks and gambler's ruin.

Today: Random Variables

1. What is a random variable?
2. Probability distribution function (PDF) and Cumulative distribution function (CDF).
3. Joint probability distribution and independent random variables
4. Important random variables
    - Bernoulli: indicator random variables.
    - Uniform: simple and powerful. An equalizing force.
    - Binomial: sum of independent indicator random variables.
    - Exponential: the waiting time to the first success.

A Random Variable is a “Measurable Property”

Temperature: “measurable property” of random positions and velocities of molecules.

A Random Variable is a “Measurable Property”

Temperature: “measurable property” of random positions and velocities of molecules.

Toss 3 coins.

$$\text{number-of-heads}(\text{HTT}) = 1;$$
$$\text{all-tosses-match}(\text{HTT}) = 0.$$

A Random Variable is a “Measurable Property”

Temperature: “measurable property” of random positions and velocities of molecules.

Toss 3 coins.

$$\text{number-of-heads}(\text{HTT}) = 1;$$
$$\text{all-tosses-match}(\text{HTT}) = 0.$$

	SAMPLE SPACE Ω							
ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

A Random Variable is a “Measurable Property”

Temperature: “measurable property” of random positions and velocities of molecules.

Toss 3 coins.

$$\text{number-of-heads(HTT)} = 1;$$
$$\text{all-tosses-match(HTT)} = 0.$$

	Sample Space Ω								
ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT	
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0	← number of heads

A Random Variable is a "Measurable Property"

Temperature: "measurable property" of random positions and velocities of molecules.

Toss 3 coins.

$$\text{number-of-heads}(\text{HTT}) = 1;$$
$$\text{all-tosses-match}(\text{HTT}) = 0.$$

	SAMPLE SPACE Ω								
ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT	
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0	← number of heads
$\mathbf{Y}(\omega)$	1	0	0	0	0	0	0	1	← matching tosses

A Random Variable is a “Measurable Property”

Temperature: “measurable property” of random positions and velocities of molecules.

Toss 3 coins.

$$\text{number-of-heads}(\text{HTT}) = 1;$$
$$\text{all-tosses-match}(\text{HTT}) = 0.$$

	Sample Space Ω								
ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT	
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0	← number of heads
$\mathbf{Y}(\omega)$	1	0	0	0	0	0	0	1	← matching tosses
$\mathbf{Z}(\omega)$	8	2	2	$\frac{1}{2}$	2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$	← H: double your money T: halve your money

A Random Variable is a "Measurable Property"

Temperature: "measurable property" of random positions and velocities of molecules.

Toss 3 coins.

$$\text{number-of-heads}(\text{HTT}) = 1;$$
$$\text{all-tosses-match}(\text{HTT}) = 0.$$

	Sample Space Ω								
ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT	
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0	← number of heads
$\mathbf{Y}(\omega)$	1	0	0	0	0	0	0	1	← matching tosses
$\mathbf{Z}(\omega)$	8	2	2	$\frac{1}{2}$	2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$	← H: double your money T: halve your money

Can use random variables to define events:

$$\{\mathbf{X} = 2\} \;=\; \{\text{HHT}, \text{HTH}, \text{THH}\} \qquad\qquad \mathbb{P}[\mathbf{X} = 2] \;=\; \tfrac{3}{8}$$

A Random Variable is a "Measurable Property"

Temperature: "measurable property" of random positions and velocities of molecules.

Toss 3 coins.

$$\text{number-of-heads(HTT)} = 1;$$
$$\text{all-tosses-match(HTT)} = 0.$$

Sample Space Ω

ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT	
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0	← number of heads
$\mathbf{Y}(\omega)$	1	0	0	0	0	0	0	1	← matching tosses
$\mathbf{Z}(\omega)$	8	2	2	$\frac{1}{2}$	2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$	← H: double your money T: halve your money

Can use random variables to define events:

$$\{\mathbf{X} = 2\} = \{\text{HHT}, \text{HTH}, \text{THH}\} \qquad \mathbb{P}[\mathbf{X} = 2] = \tfrac{3}{8}$$
$$\{\mathbf{X} \geq 2\} = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{THH}\} \qquad \mathbb{P}[\mathbf{X} \geq 2] = \tfrac{1}{2}$$

A Random Variable is a "Measurable Property"

Temperature: "measurable property" of random positions and velocities of molecules.

Toss 3 coins.

$$\text{number-of-heads}(\text{HTT}) = 1;$$
$$\text{all-tosses-match}(\text{HTT}) = 0.$$

	Sample Space Ω								
ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT	
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0	← number of heads
$\mathbf{Y}(\omega)$	1	0	0	0	0	0	0	1	← matching tosses
$\mathbf{Z}(\omega)$	8	2	2	$\frac{1}{2}$	2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$	← H: double your money T: halve your money

Can use random variables to define events:

$$\{\mathbf{X} = 2\} = \{\text{HHT}, \text{HTH}, \text{THH}\} \qquad \mathbb{P}[\mathbf{X} = 2] = \tfrac{3}{8}$$
$$\{\mathbf{X} \geq 2\} = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{THH}\} \qquad \mathbb{P}[\mathbf{X} \geq 2] = \tfrac{1}{2}$$
$$\{\mathbf{Y} = 1\} = \{\text{HHH}, \text{TTT}\} \qquad \mathbb{P}[\mathbf{Y} = 1] = \tfrac{1}{4}$$

A Random Variable is a "Measurable Property"

Temperature: "measurable property" of random positions and velocities of molecules.

Toss 3 coins.

$$\text{number-of-heads}(\text{HTT}) = 1;$$
$$\text{all-tosses-match}(\text{HTT}) = 0.$$

	SAMPLE SPACE Ω								
ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT	
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0	← number of heads
$\mathbf{Y}(\omega)$	1	0	0	0	0	0	0	1	← matching tosses
$\mathbf{Z}(\omega)$	8	2	2	$\frac{1}{2}$	2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$	← H: double your money T: halve your money

Can use random variables to define events:

$$\begin{aligned}
\{\mathbf{X}=2\} &= \{\text{HHT}, \text{HTH}, \text{THH}\} & \mathbb{P}[\mathbf{X}=2] &= \tfrac{3}{8} \\
\{\mathbf{X}\geq 2\} &= \{\text{HHH}, \text{HHT}, \text{HTH}, \text{THH}\} & \mathbb{P}[\mathbf{X}\geq 2] &= \tfrac{1}{2} \\
\{\mathbf{Y}=1\} &= \{\text{HHH}, \text{TTT}\} & \mathbb{P}[\mathbf{Y}=1] &= \tfrac{1}{4} \\
\{\mathbf{X}\geq 2 \text{ AND } \mathbf{Y}=1\} &= \{\text{HHH}\} & \mathbb{P}[\mathbf{X}\geq 2 \text{ AND } \mathbf{Y}=1] &= \tfrac{1}{8}
\end{aligned}$$

Probability Distribution Function (PDF)

$$\underset{\Omega}{\{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{HTT}, \mathrm{THH}, \mathrm{THT}, \mathrm{TTH}, \mathrm{TTT}\}} \xrightarrow{\mathbf{X}} \underset{\mathbf{X}(\Omega)}{\{3, 2, 1, 0\}}$$

Probability Distribution Function (PDF)

$$\underbrace{\{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTH}, \text{TTT}\}}_{\Omega} \xrightarrow{\mathbf{X}} \underbrace{\{3, 2, 1, 0\}}_{\mathbf{X}(\Omega)}$$

Each *possible* value x of the random variable $\mathbf{X}$ corresponds to an event,

x	0	1	2	3
Event	{TTT}			

Probability Distribution Function (PDF)

$$\underset{\Omega}{\{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTH}, \text{TTT}\}} \xrightarrow{\mathbf{X}} \underset{\mathbf{X}(\Omega)}{\{3, 2, 1, 0\}}$$

Each *possible* value x of the random variable $\mathbf{X}$ corresponds to an event,

x	0	1	2	3
Event	{TTT}	{HTT, THT, TTH}	{HHT, HTH, THH}	{HHH}

Probability Distribution Function (PDF)

$$\underset{\Omega}{\{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTH}, \text{TTT}\}} \xrightarrow{\mathbf{X}} \underset{\mathbf{X}(\Omega)}{\{3, 2, 1, 0\}}$$

Each *possible* value x of the random variable $\mathbf{X}$ corresponds to an event,

x	0	1	2	3
Event	$\{\text{TTT}\}$	$\{\text{HTT}, \text{THT}, \text{TTH}\}$	$\{\text{HHT}, \text{HTH}, \text{THH}\}$	$\{\text{HHH}\}$

For each $x \in \mathbf{X}(\Omega)$, compute $\mathbb{P}[\mathbf{X} = x]$ by adding the outcome-probabilities,

	possible values $x \in \mathbf{X}(\Omega)$			
x	0	1	2	3
$P_{\mathbf{X}}(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Probability Distribution Function (PDF)

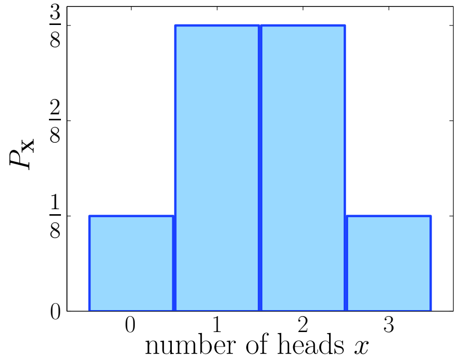
$$\underset{\Omega}{\{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{HTT}, \mathrm{THH}, \mathrm{THT}, \mathrm{TTH}, \mathrm{TTT}\}} \xrightarrow{\mathbf{X}} \underset{\mathbf{X}(\Omega)}{\{3, 2, 1, 0\}}$$

Each *possible* value x of the random variable $\mathbf{X}$ corresponds to an event,

x	0	1	2	3
Event	$\{\mathrm{TTT}\}$	$\{\mathrm{HTT}, \mathrm{THT}, \mathrm{TTH}\}$	$\{\mathrm{HHT}, \mathrm{HTH}, \mathrm{THH}\}$	$\{\mathrm{HHH}\}$

For each $x \in \mathbf{X}(\Omega)$, compute $\mathbb{P}[\mathbf{X} = x]$ by adding the outcome-probabilities,

	possible values $x \in \mathbf{X}(\Omega)$			
x	0	1	2	3
$P_{\mathbf{X}}(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$



Probability Distribution Function (PDF)

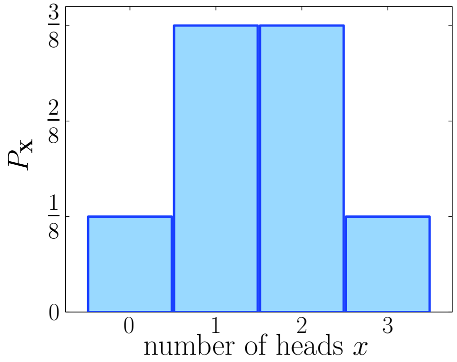
$$\underset{\Omega}{\{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTH}, \text{TTT}\}} \xrightarrow{\mathbf{X}} \underset{\mathbf{X}(\Omega)}{\{3, 2, 1, 0\}}$$

Each *possible* value x of the random variable $\mathbf{X}$ corresponds to an event,

x	0	1	2	3
Event	{TTT}	{HTT, THT, TTH}	{HHT, HTH, THH}	{HHH}

For each $x \in \mathbf{X}(\Omega)$, compute $\mathbb{P}[\mathbf{X} = x]$ by adding the outcome-probabilities,

	possible values $x \in \mathbf{X}(\Omega)$			
x	0	1	2	3
$P_{\mathbf{X}}(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$



Probability Distribution Function (PDF). The probability distribution function $P_{\mathbf{X}}(x)$ is the probability for the random variable $\mathbf{X}$ to take value x,

$$P_{\mathbf{X}}(x) = \mathbb{P}[\mathbf{X} = x].$$

PDF for the Sum of Two Dice

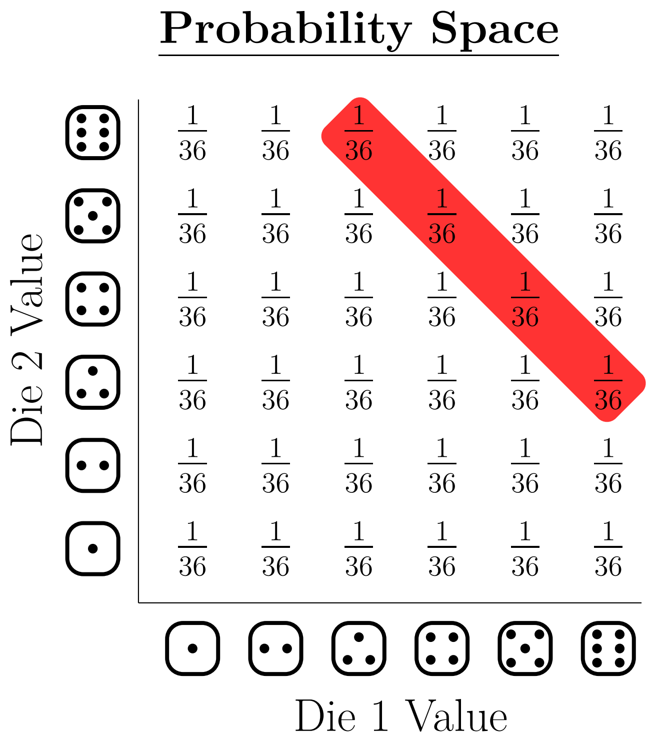
Probability Space

Die 2 Value

⚅	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚄	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚃	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚂	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚁	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚀	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
	⚀	⚁	⚂	⚃	⚄	⚅

Die 1 Value

PDF for the Sum of Two Dice



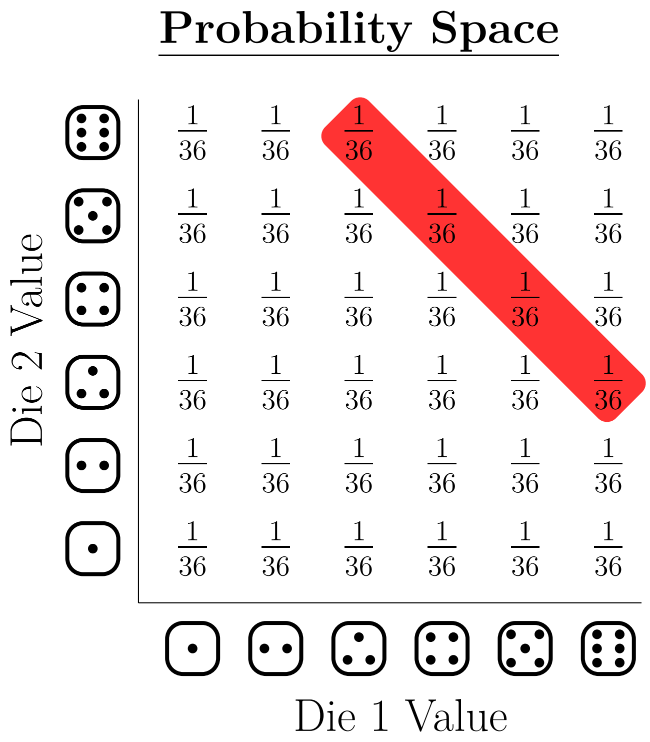
Probability Space

⚅	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚄	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚃	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚂	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚁	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚀	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
	⚀	⚁	⚂	⚃	⚄	⚅

$\mathbf{X} = 9$ has four outcomes,

$$\mathbb{P}[\mathbf{X} = 9] = 4 \times \tfrac{1}{36} = \tfrac{1}{9}.$$

PDF for the Sum of Two Dice



$\mathbf{X}=9$ has four outcomes,

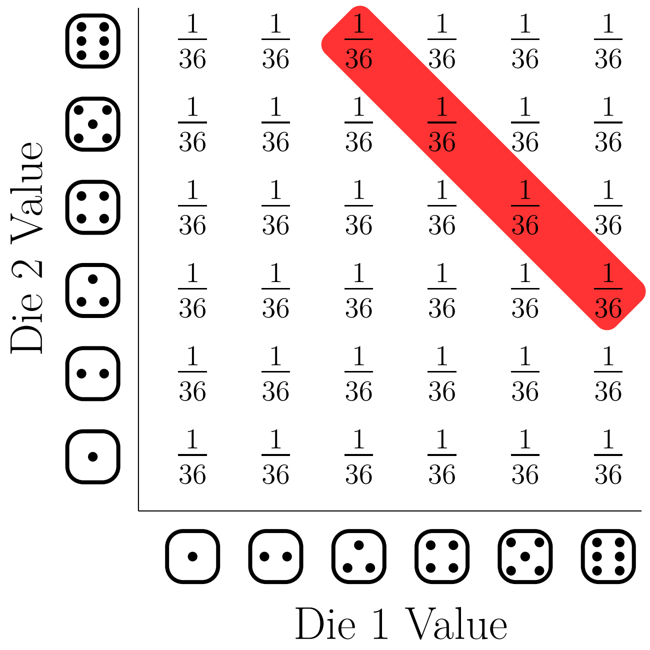
$$\mathbb{P}[\mathbf{X}=9]=4\times\tfrac{1}{36}=\tfrac{1}{9}.$$

Possible sums are $\mathbf{X}\in\{2,3,\ldots,12\}$ and the PDF is

x	2	3	4	5	6	7	8	9	10	11	12
$P_{\mathbf{X}}(x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

PDF for the Sum of Two Dice

Probability Space



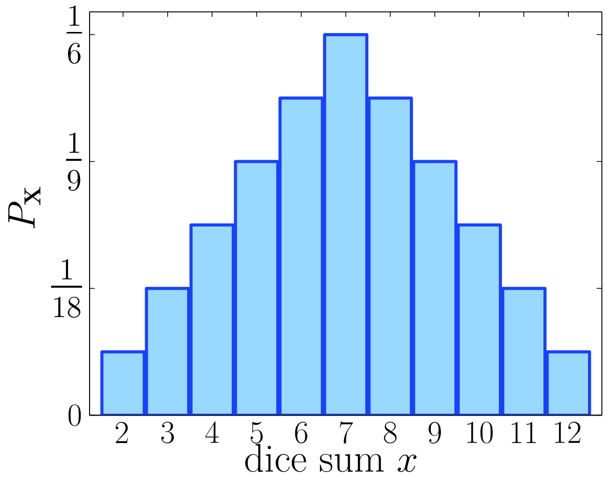
⚅	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚄	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚃	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚂	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚁	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
⚀	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
	⚀	⚁	⚂	⚃	⚄	⚅

$\mathbf{X} = 9$ has four outcomes,

$$\mathbb{P}[\mathbf{X} = 9] = 4 \times \tfrac{1}{36} = \tfrac{1}{9}.$$

Possible sums are $\mathbf{X} \in \{2, 3, \ldots, 12\}$ and the PDF is

x	2	3	4	5	6	7	8	9	10	11	12
$P_{\mathbf{X}}(x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$



Joint PDF: More Than One Random Variable

	Sample Space Ω								
ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT	
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0	← number of heads
$\mathbf{Y}(\omega)$	1	0	0	0	0	0	0	1	← matching tosses

$$\begin{aligned} \mathbb{P}[\mathbf{X}=0,\mathbf{Y}=0] &= 0 \\ \mathbb{P}[\mathbf{X}=1,\mathbf{Y}=0] &= \tfrac{3}{8}. \end{aligned}$$

Joint PDF: More Than One Random Variable

SAMPLE SPACE Ω

ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT	
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0	← number of heads
$\mathbf{Y}(\omega)$	1	0	0	0	0	0	0	1	← matching tosses

$$\begin{aligned}\mathbb{P}[\mathbf{X}=0,\mathbf{Y}=0] &= 0\\ \mathbb{P}[\mathbf{X}=1,\mathbf{Y}=0] &= \tfrac{3}{8}.\end{aligned}$$

$$P_{\mathbf{XY}}(x,y)=\mathbb{P}[\mathbf{X}=x,\mathbf{Y}=y].$$

		X				
$P_{\mathbf{XY}}(x,y)$		0	1	2	3	row sums
Y	0	0	$\frac{3}{8}$	$\frac{3}{8}$	0	$\mathbf{\frac{3}{4}}$
	1	$\frac{1}{8}$	0	0	$\frac{1}{8}$	$\mathbf{\frac{1}{4}}$
column sums		$\mathbf{\frac{1}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{1}{8}}$	

$$P_{\mathbf{Y}}(y)=\sum_{x\in\mathbf{X}(\Omega)}P_{\mathbf{XY}}(x,y)$$

$$P_{\mathbf{X}}(x)=\sum_{y\in\mathbf{Y}(\Omega)}P_{\mathbf{XY}}(x,y)$$

Joint PDF: More Than One Random Variable

	Sample Space Ω								
ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT	
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0	← number of heads
$\mathbf{Y}(\omega)$	1	0	0	0	0	0	0	1	← matching tosses

$$\begin{aligned}\mathbb{P}[\mathbf{X}=0,\mathbf{Y}=0] &= 0\\ \mathbb{P}[\mathbf{X}=1,\mathbf{Y}=0] &= \tfrac{3}{8}.\end{aligned}$$

$$P_{\mathbf{XY}}(x,y)=\mathbb{P}[\mathbf{X}=x,\mathbf{Y}=y].$$

$$\mathbb{P}[\mathbf{X}+\mathbf{Y}\le 2]=0+\tfrac{3}{8}+\tfrac{3}{8}+\tfrac{1}{8}+0=\tfrac{7}{8}.$$

		X				
$P_{\mathbf{XY}}(x,y)$		0	1	2	3	row sums
Y	0	0	$\frac{3}{8}$	$\frac{3}{8}$	0	$\mathbf{\frac{3}{4}}$
	1	$\frac{1}{8}$	0	0	$\frac{1}{8}$	$\mathbf{\frac{1}{4}}$
column sums		$\mathbf{\frac{1}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{1}{8}}$	

$$P_{\mathbf{Y}}(y)=\sum_{x\in\mathbf{X}(\Omega)}P_{\mathbf{XY}}(x,y)$$

$$P_{\mathbf{X}}(x)=\sum_{y\in\mathbf{Y}(\Omega)}P_{\mathbf{XY}}(x,y)$$

Joint PDF: More Than One Random Variable

	Sample Space Ω								
ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT	
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0	← number of heads
$\mathbf{Y}(\omega)$	1	0	0	0	0	0	0	1	← matching tosses

$$\begin{aligned}\mathbb{P}[\mathbf{X}=0,\mathbf{Y}=0] &= 0\\ \mathbb{P}[\mathbf{X}=1,\mathbf{Y}=0] &= \tfrac{3}{8}.\end{aligned}$$

$$P_{\mathbf{XY}}(x,y)=\mathbb{P}[\mathbf{X}=x,\mathbf{Y}=y].$$

$$\mathbb{P}[\mathbf{X}+\mathbf{Y}\le 2]=0+\tfrac{3}{8}+\tfrac{3}{8}+\tfrac{1}{8}+0=\tfrac{7}{8}.$$

$$\mathbb{P}[\mathbf{Y}=1 \text{ AND } \mathbf{X}+\mathbf{Y}\le 2]=\tfrac{1}{8}+0=\tfrac{1}{8}.$$

X

$P_{\mathbf{XY}}(x,y)$

		0	1	2	3	row sums
Y	0	0	$\frac{3}{8}$	$\frac{3}{8}$	0	$\mathbf{\frac{3}{4}}$
	1	$\frac{1}{8}$	0	0	$\frac{1}{8}$	$\mathbf{\frac{1}{4}}$
column sums		$\mathbf{\frac{1}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{1}{8}}$	

$$P_{\mathbf{Y}}(y)=\sum_{x\in\mathbf{X}(\Omega)}P_{\mathbf{XY}}(x,y)$$

$$P_{\mathbf{X}}(x)=\sum_{y\in\mathbf{Y}(\Omega)}P_{\mathbf{XY}}(x,y)$$

Joint PDF: More Than One Random Variable

	Sample Space Ω								
ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT	
$P(\omega)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	
$\mathbf{X}(\omega)$	3	2	2	1	2	1	1	0	← number of heads
$\mathbf{Y}(\omega)$	1	0	0	0	0	0	0	1	← matching tosses

$$\begin{aligned}\mathbb{P}[\mathbf{X}=0,\mathbf{Y}=0] &= 0\\ \mathbb{P}[\mathbf{X}=1,\mathbf{Y}=0] &= \tfrac{3}{8}.\end{aligned}$$

$$P_{\mathbf{XY}}(x,y)=\mathbb{P}[\mathbf{X}=x,\mathbf{Y}=y].$$

$$\mathbb{P}[\mathbf{X}+\mathbf{Y}\le 2]=0+\tfrac{3}{8}+\tfrac{3}{8}+\tfrac{1}{8}+0=\tfrac{7}{8}.$$

$$\mathbb{P}[\mathbf{Y}=1\text{ AND }\mathbf{X}+\mathbf{Y}\le 2]=\tfrac{1}{8}+0=\tfrac{1}{8}.$$

$$\begin{aligned}\mathbb{P}[\mathbf{Y}=1\mid\mathbf{X}+\mathbf{Y}\le 2] &= \frac{[\mathbf{Y}=1\text{ AND }\mathbf{X}+\mathbf{Y}\le 2]}{\mathbb{P}[\mathbf{X}+\mathbf{Y}\le 2]}\\ &= \tfrac{1}{8}/\tfrac{7}{8}=\tfrac{1}{7}\end{aligned}$$

$P_{\mathbf{XY}}(x,y)$		$\mathbf{X}$				
		0	1	2	3	row sums
$\mathbf{Y}$	0	0	$\frac{3}{8}$	$\frac{3}{8}$	0	$\mathbf{\frac{3}{4}}$
	1	$\frac{1}{8}$	0	0	$\frac{1}{8}$	$\mathbf{\frac{1}{4}}$
column sums		$\mathbf{\frac{1}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{1}{8}}$	

$$P_{\mathbf{Y}}(y)=\sum_{x\in\mathbf{X}(\Omega)}P_{\mathbf{XY}}(x,y)$$

$$P_{\mathbf{X}}(x)=\sum_{y\in\mathbf{Y}(\Omega)}P_{\mathbf{XY}}(x,y)$$

Independent Random Variables

Independent Random Variables measure unrelated quantities.
The joint-PDF is *always* the product of the marginals.

$$P_{\mathbf{XY}}(x, y) = P_{\mathbf{X}}(x)P_{\mathbf{Y}}(y) \qquad \text{for all } (x, y) \in \mathbf{X}(\Omega) \times \mathbf{Y}(\Omega).$$

Independent Random Variables

Independent Random Variables measure unrelated quantities.
The joint-PDF is *always* the product of the marginals.

$$P_{\mathbf{XY}}(x,y) = P_{\mathbf{X}}(x)P_{\mathbf{Y}}(y) \qquad \text{for all } (x,y) \in \mathbf{X}(\Omega) \times \mathbf{Y}(\Omega).$$

Our **X** and **Y** are *not* independent,

$P_{\mathbf{XY}}(x,y)$		**X**				
		0	1	2	3	
Y	0	0	$\frac{3}{8}$	$\frac{3}{8}$	0	$\mathbf{\frac{3}{4}}$
	1	$\frac{1}{8}$	0	0	$\frac{1}{8}$	$\mathbf{\frac{1}{4}}$
		$\mathbf{\frac{1}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{1}{8}}$	

$P_{\mathbf{X}}(x)P_{\mathbf{Y}}(y)$		**X**				
		0	1	2	3	
Y	0	$\frac{3}{32}$	$\frac{9}{32}$	$\frac{9}{32}$	$\frac{9}{32}$	$\mathbf{\frac{3}{4}}$
	1	$\frac{1}{32}$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{1}{32}$	$\mathbf{\frac{1}{4}}$
		$\mathbf{\frac{1}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{1}{8}}$	

Practice: Exercise 18.4, Pop Quizzes 18.5, 18.6.

Cumulative Distribution Function (CDF)

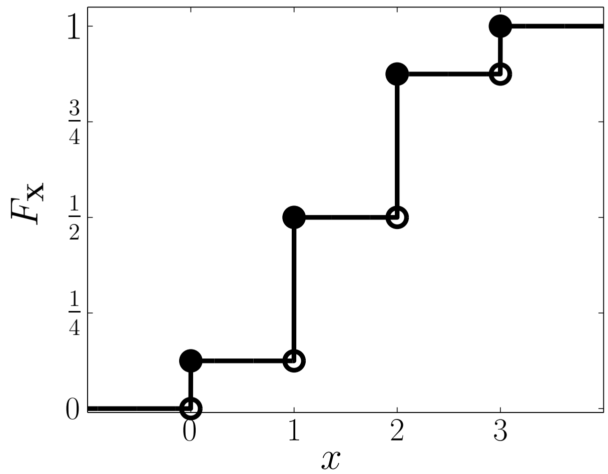
x	0	1	2	3
$P_{\mathbf{X}}(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Cumulative Distribution Function (CDF)

x	0	1	2	3
$P_{\mathbf{X}}(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$\mathbb{P}[\mathbf{X} \le x]$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	$\frac{8}{8}$

Cumulative Distribution Function (CDF)

x	0	1	2	3
$P_{\mathbf{X}}(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$\mathbb{P}[\mathbf{X} \leq x]$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	$\frac{8}{8}$



Cumulative Distribution Function (CDF). The cumulative distribution function $F_{\mathbf{X}}(x)$ is the probability for the random variable $\mathbf{X}$ to be at most x,

$$F_{\mathbf{X}}(x) = \mathbb{P}[\mathbf{X} \leq x].$$

Bernoulli Random Variable: Binary Measurable $(0, 1)$

Two outcomes: coin toss, drunk steps left or right, etc. $\mathbf{X}$ *indicates* which outcome,

$$\mathbf{X} = \begin{cases} 1 & \text{with probability } p; \\ 0 & \text{with probability } 1 - p. \end{cases}$$

Bernoulli Random Variable: Binary Measurable $(0, 1)$

Two outcomes: coin toss, drunk steps left or right, etc. $\mathbf{X}$ *indicates* which outcome,

$$\mathbf{X} = \begin{cases} 1 & \text{with probability } p; \\ 0 & \text{with probability } 1 - p. \end{cases}$$

Can add Bernoullis. Toss n independent coins. $\mathbf{X}$ is the number of H.

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n.$$

$\mathbf{X}$ is a sum of Bernoullis, each $\mathbf{X}_i$ is an independent Bernoulli.

Bernoulli Random Variable: Binary Measurable $(0, 1)$

Two outcomes: coin toss, drunk steps left or right, etc. $\mathbf{X}$ *indicates* which outcome,

$$\mathbf{X} = \begin{cases} 1 & \text{with probability } p; \\ 0 & \text{with probability } 1-p. \end{cases}$$

Can add Bernoullis. Toss n independent coins. $\mathbf{X}$ is the number of H.

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n.$$

$\mathbf{X}$ is a sum of Bernoullis, each $\mathbf{X}_i$ is an independent Bernoulli.

Drunk makes n steps. Let $\mathbf{R}$ be the number of right steps

$$\mathbf{R} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n.$$

$\mathbf{R}$ is a sum of Bernoullis.

Bernoulli Random Variable: Binary Measurable $(0, 1)$

Two outcomes: coin toss, drunk steps left or right, etc. $\mathbf{X}$ *indicates* which outcome,

$$\mathbf{X} = \begin{cases} 1 & \text{with probability } p; \\ 0 & \text{with probability } 1 - p. \end{cases}$$

Can add Bernoullis. Toss n independent coins. $\mathbf{X}$ is the number of H.

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n.$$

$\mathbf{X}$ is a sum of Bernoullis, each $\mathbf{X}_i$ is an independent Bernoulli.

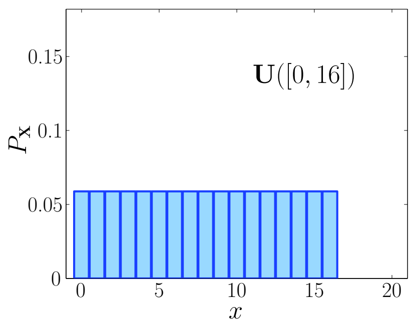
Drunk makes n steps. Let $\mathbf{R}$ be the number of right steps

$$\mathbf{R} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n.$$

$\mathbf{R}$ is a sum of Bernoullis. $\mathbf{L} = n - \mathbf{R}$ and the final position $\mathbf{X}$ is:

$$\mathbf{X} = \mathbf{R} - \mathbf{L} = 2\mathbf{R} - n = 2(\mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n) - n.$$

Uniform Random Variable: Every Value Equally Likely

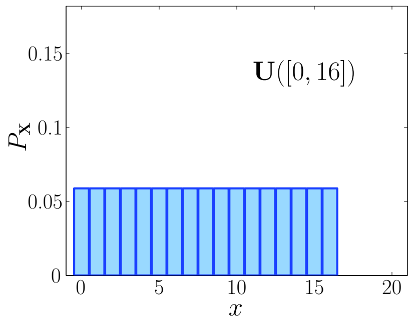


n possible values $\{1,2,\ldots,n\}$, each with probability $\frac{1}{n}$:

$$P_{\mathbf{X}}(k) = \tfrac{1}{n}, \qquad\qquad \text{for } k = 1,\ldots,n.$$

Roll of a 6-sided *fair* die $\sim \mathbf{U}[6]$. (Uniform on $\{1,\ldots,6\}$)

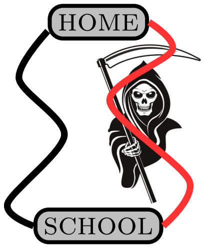
Uniform Random Variable: Every Value Equally Likely



n possible values $\{1, 2, \ldots, n\}$, each with probability $\frac{1}{n}$:

$$P_{\mathbf{X}}(k) = \frac{1}{n}, \qquad \text{for } k = 1, \ldots, n.$$

Roll of a 6-sided *fair* die $\sim \mathbf{U}[6]$. (Uniform on $\{1, \ldots, 6\}$)



Example: Matching game (uniform is an equalizer in games of strategy).

GR will pick a path to relieve you of your lunch money.

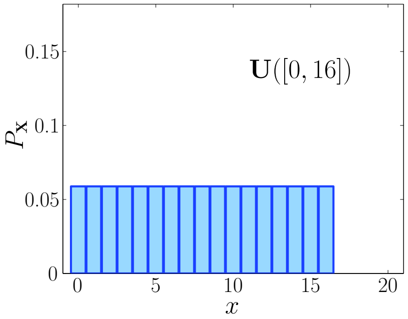
If you pick your path uniformly, you win half the time.

Uniform Random Variable: Every Value Equally Likely

n possible values $\{1, 2, \ldots, n\}$, each with probability $\frac{1}{n}$:

$$P_{\mathbf{X}}(k) = \frac{1}{n}, \qquad \text{for } k = 1, \ldots, n.$$

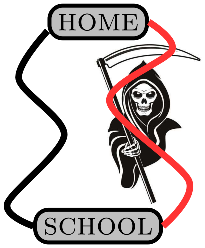
Roll of a 6-sided *fair* die $\sim \mathbf{U}[6]$. (Uniform on $\{1, \ldots, 6\}$)



Example: Matching game (uniform is an equalizer in games of strategy).

GR will pick a path to relieve you of your lunch money.

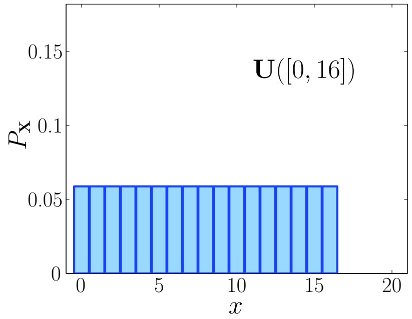
If you pick your path uniformly, you win half the time.



Example 18.2: Guessing Larger or Smaller

I pick two numbers from $\{1, \ldots, 5\}$, as I please. I *randomly* show you one of the two, x.

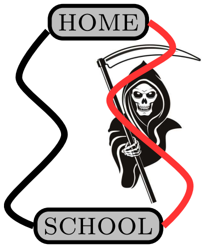
Uniform Random Variable: Every Value Equally Likely



n possible values $\{1, 2, \ldots, n\}$, each with probability $\frac{1}{n}$:

$$P_{\mathbf{X}}(k) = \frac{1}{n}, \qquad\qquad \text{for } k = 1, \ldots, n.$$

Roll of a 6-sided *fair* die $\sim \mathbf{U}[6]$. (Uniform on $\{1, \ldots, 6\}$)



Example: Matching game (uniform is an equalizer in games of strategy).

GR will pick a path to relieve you of your lunch money.

If you pick your path uniformly, you win half the time.

Example 18.2: Guessing Larger or Smaller

I pick two numbers from $\{1, \ldots, 5\}$, as I please. I *randomly* show you one of the two, x.

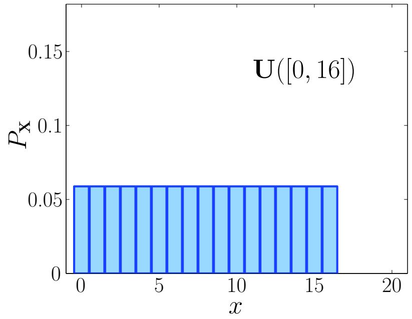
You must guess if x is the larger or smaller of my two numbers.

Uniform Random Variable: Every Value Equally Likely

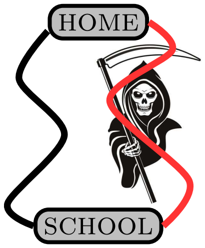
n possible values $\{1, 2, \ldots, n\}$, each with probability $\frac{1}{n}$:

$$P_{\mathbf{X}}(k) = \frac{1}{n}, \qquad \text{for } k = 1, \ldots, n.$$

Roll of a 6-sided *fair* die $\sim \mathbf{U}[6]$. (Uniform on $\{1, \ldots, 6\}$)



Example: Matching game (uniform is an equalizer in games of strategy).
GR will pick a path to relieve you of your lunch money.
If you pick your path uniformly, you win half the time.



Example 18.2: Guessing Larger or Smaller

I pick two numbers from $\{1, \ldots, 5\}$, as I please. I *randomly* show you one of the two, x.
You must guess if x is the larger or smaller of my two numbers.

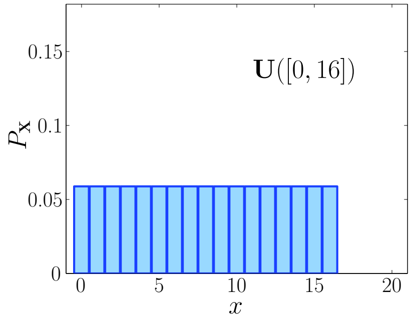
You always say smaller: you win $\frac{1}{2}$ the time.

Uniform Random Variable: Every Value Equally Likely

n possible values $\{1,2,\ldots,n\}$, each with probability $\frac{1}{n}$:

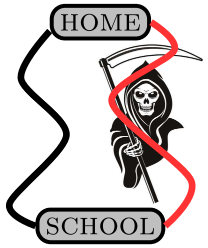
$$P_{\mathbf{X}}(k) = \frac{1}{n}, \qquad \text{for } k = 1,\ldots,n.$$

Roll of a 6-sided *fair* die $\sim \mathbf{U}[6]$. (Uniform on $\{1,\ldots,6\}$)



Example: Matching game (uniform is an equalizer in games of strategy).
GR will pick a path to relieve you of your lunch money.

If you pick your path uniformly, you win half the time.



Example 18.2: Guessing Larger or Smaller

I pick two numbers from $\{1,\ldots,5\}$, as I please. I *randomly* show you one of the two, x.
You must guess if x is the larger or smaller of my two numbers.

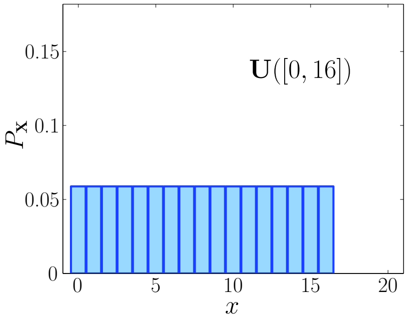
You always say smaller: you win $\frac{1}{2}$ the time.
You say smaller if $x \le 3$ and larger if $x > 3$. I pick numbers 1,2: you win $\frac{1}{2}$ the time.

Uniform Random Variable: Every Value Equally Likely

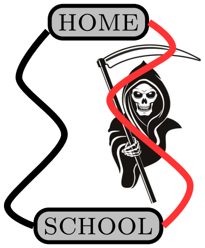
n possible values $\{1, 2, \ldots, n\}$, each with probability $\frac{1}{n}$:

$$P_{\mathbf{X}}(k) = \frac{1}{n}, \qquad \text{for } k = 1, \ldots, n.$$

Roll of a 6-sided *fair* die $\sim$ $\mathbf{U}[6]$. (Uniform on $\{1, \ldots, 6\}$)



Example: Matching game (uniform is an equalizer in games of strategy).
GR will pick a path to relieve you of your lunch money.
If you pick your path uniformly, you win half the time.



Example 18.2: Guessing Larger or Smaller

I pick two numbers from $\{1, \ldots, 5\}$, as I please. I *randomly* show you one of the two, x.
You must guess if x is the larger or smaller of my two numbers.

You always say smaller: you win $\frac{1}{2}$ the time.
You say smaller if $x \leq 3$ and larger if $x > 3$. I pick numbers 1,2: you win $\frac{1}{2}$ the time.

You have a strategy which wins *more* than $\frac{1}{2}$ the time, and I *cannot* prevent it!

Binomial Random Variable: Sum of Bernoullis

$\mathbf{X}$ = number of heads in n independent coin tosses with probability p of heads,

$$\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n.$$

← sum of n independent Bernoullis, $\mathbf{X}_i \sim \text{Bernoulli}(p)$

Binomial Random Variable: Sum of Bernoullis

$\mathbf{X}$ = number of heads in n independent coin tosses with probability p of heads,

$$\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n.$$

← sum of n independent Bernoullis, $\mathbf{X}_i \sim \text{Bernoulli}(p)$

n=5,$\mathbf{X}$=3:

HHHTT	HHTTH	HTTHH	TTHHH	HHTHT
HTHTH	THTHH	HTHHT	THHTH	THHHT

Binomial Random Variable: Sum of Bernoullis

$\mathbf{X}$ = number of heads in n independent coin tosses with probability p of heads,

$$\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n.$$

← sum of n independent Bernoullis, $\mathbf{X}_i \sim \text{Bernoulli}(p)$

n=5,$\mathbf{X}$=3:

HHHTT	HHTTH	HTTHH	TTHHH	HHTHT
HTHTH	THTHH	HTHHT	THHTH	THHHT

← each has probability $p^3(1-p)^2$ (independence)

Binomial Random Variable: Sum of Bernoullis

$\mathbf{X}$ = number of heads in n independent coin tosses with probability p of heads,

$$\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n.$$

← sum of n independent Bernoullis, $\mathbf{X}_i \sim \text{Bernoulli}(p)$

n=5,$\mathbf{X}$=3:

HHHTT HHTTH HTTHH TTHHH HHTHT
HTHTH THTHH HTHHT THHTH THHHT

← each has probability $p^3(1-p)^2$ (independence)

$$\mathbb{P}[\mathbf{X} = 3 \mid n = 5] = 10p^3(1-p)^5$$

← add outcome probabilities

Binomial Random Variable: Sum of Bernoullis

$\mathbf{X}$ = number of heads in n independent coin tosses with probability p of heads,

$$\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n.$$

← sum of n independent Bernoullis, $\mathbf{X}_i \sim \text{Bernoulli}(p)$

n=5,$\mathbf{X}$=3:

HHHTT HHTTH HTTHH TTHHH HHTHT
HTHTH THTHH HTHHT THHTH THHHT

← each has probability $p^3(1-p)^2$ (independence)

$$\mathbb{P}[\mathbf{X} = 3 \mid n = 5] = 10p^3(1-p)^5$$

← add outcome probabilities

In general, $\binom{n}{k}$ sequences with k heads.

Binomial Random Variable: Sum of Bernoullis

$\mathbf{X}$ = number of heads in n independent coin tosses with probability p of heads,

$$\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n.$$

← sum of n independent Bernoullis, $\mathbf{X}_i \sim \text{Bernoulli}(p)$

n=5,$\mathbf{X}$=3:

HHHTT HHTTH HTTHH TTHHH HHTHT
HTHTH THTHH HTHHT THHTH THHHT

← each has probability $p^3(1-p)^2$ (independence)

$$\mathbb{P}[\mathbf{X} = 3 \mid n = 5] = 10p^3(1-p)^5$$

← add outcome probabilities

In general, $\binom{n}{k}$ sequences with k heads.
Each has probability $p^k(1-p)^{n-k}$

Binomial Random Variable: Sum of Bernoullis

$\mathbf{X}$ = number of heads in n independent coin tosses with probability p of heads,

$$\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n.$$

← sum of n independent Bernoullis, $\mathbf{X}_i \sim \text{Bernoulli}(p)$

n=5,$\mathbf{X}$=3:

HHHTT HHTTH HTTHH TTHHH HHTHT
HTHTH THTHH HTHHT THHTH THHHT

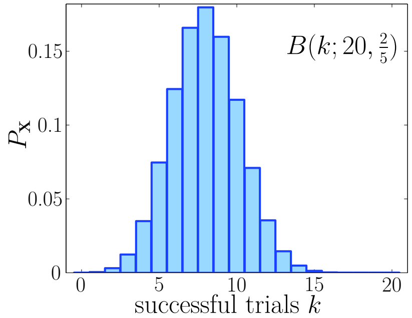
← each has probability $p^3(1-p)^2$ (independence)

$$\mathbb{P}[\mathbf{X} = 3 \mid n = 5] = 10p^3(1-p)^5$$

← add outcome probabilities

In general, $\binom{n}{k}$ sequences with k heads.
Each has probability $p^k(1-p)^{n-k}$, so

$$\mathbb{P}[\mathbf{X} = k \mid n] = \binom{n}{k} p^k(1-p)^{n-k}.$$



Binomial Distribution. $\mathbf{X}$ is the number of successes in n independent trials with success probability p on each trial: $\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n$ where $\mathbf{X}_i \sim \text{Bernoulli}(p)$.

$$P_{\mathbf{X}}(k) = B(k; n, p) = \binom{n}{k} p^k(1-p)^{n-k}.$$

Binomial Random Variable: Sum of Bernoullis

$\mathbf{X}$ = number of heads in n independent coin tosses with probability p of heads,

$$\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n.$$

← sum of n independent Bernoullis, $\mathbf{X}_i \sim \text{Bernoulli}(p)$

n=5,$\mathbf{X}$=3:

HHHTT HHTTH HTTHH TTHHH HHTHT
HTHTH THTHH HTHHT THHTH THHHT

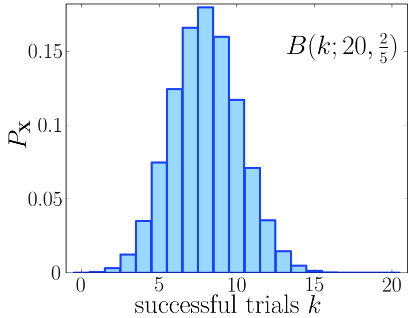
← each has probability $p^3(1-p)^2$ (independence)

$$\mathbb{P}[\mathbf{X} = 3 \mid n = 5] = 10p^3(1-p)^5$$

← add outcome probabilities

In general, $\binom{n}{k}$ sequences with k heads.
Each has probability $p^k(1-p)^{n-k}$, so

$$\mathbb{P}[\mathbf{X} = k \mid n] = \binom{n}{k} p^k (1-p)^{n-k}.$$



Binomial Distribution. $\mathbf{X}$ is the number of successes in n independent trials with success probability p on each trial: $\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n$ where $\mathbf{X}_i \sim \text{Bernoulli}(p)$.

$$P_{\mathbf{X}}(k) = B(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Example: guessing correctly on the multiple choice quiz: $n = 15$ questions, 5 choices ($p = \frac{1}{5}$).

Binomial Random Variable: Sum of Bernoullis

$\mathbf{X}$ = number of heads in n independent coin tosses with probability p of heads,

$$\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n.$$

← sum of n independent Bernoullis, $\mathbf{X}_i \sim \text{Bernoulli}(p)$

n=5,$\mathbf{X}$=3:

HHHTT HHTTH HTTHH TTHHH HHTHT
HTHTH THTHH HTHHT THHTH THHHT

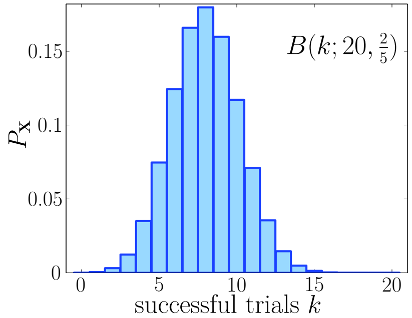
← each has probability $p^3(1-p)^2$ (independence)

$$\mathbb{P}[\mathbf{X} = 3 \mid n = 5] = 10p^3(1-p)^5$$

← add outcome probabilities

In general, $\binom{n}{k}$ sequences with k heads.
Each has probability $p^k(1-p)^{n-k}$, so

$$\mathbb{P}[\mathbf{X} = k \mid n] = \binom{n}{k} p^k(1-p)^{n-k}.$$



Binomial Distribution. $\mathbf{X}$ is the number of successes in n independent trials with success probability p on each trial: $\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n$ where $\mathbf{X}_i \sim \text{Bernoulli}(p)$.

$$P_{\mathbf{X}}(k) = B(k; n, p) = \binom{n}{k} p^k(1-p)^{n-k}.$$

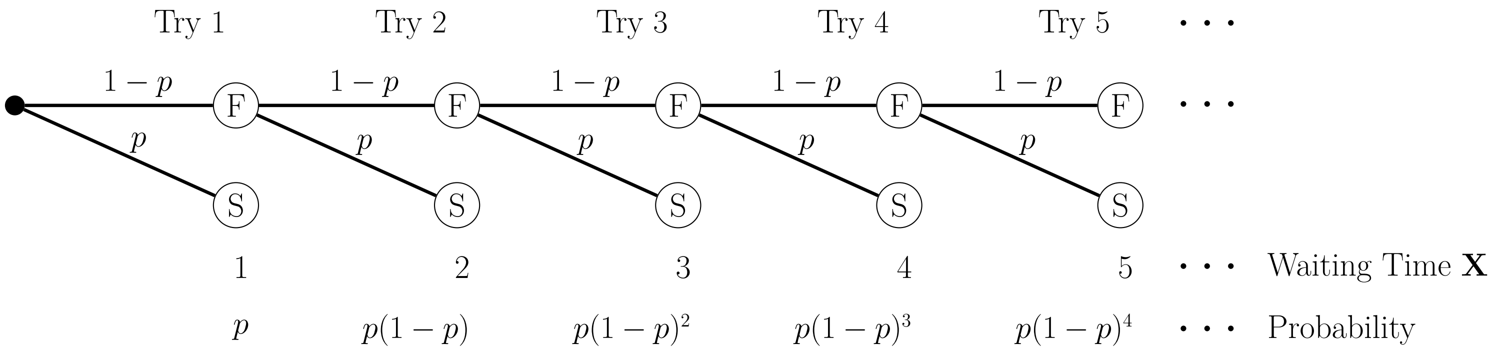
Example: guessing correctly on the multiple choice quiz: $n = 15$ questions, 5 choices ($p = \frac{1}{5}$).

number correct, k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
probability	0.035	0.132	0.231	0.250	0.188	0.103	0.043	0.014	0.003	7×10^{-4}	10^{-4}	10^{-5}	10^{-6}	~ 0	~ 0	~ 0

chances of passing are ≈ 0.4%

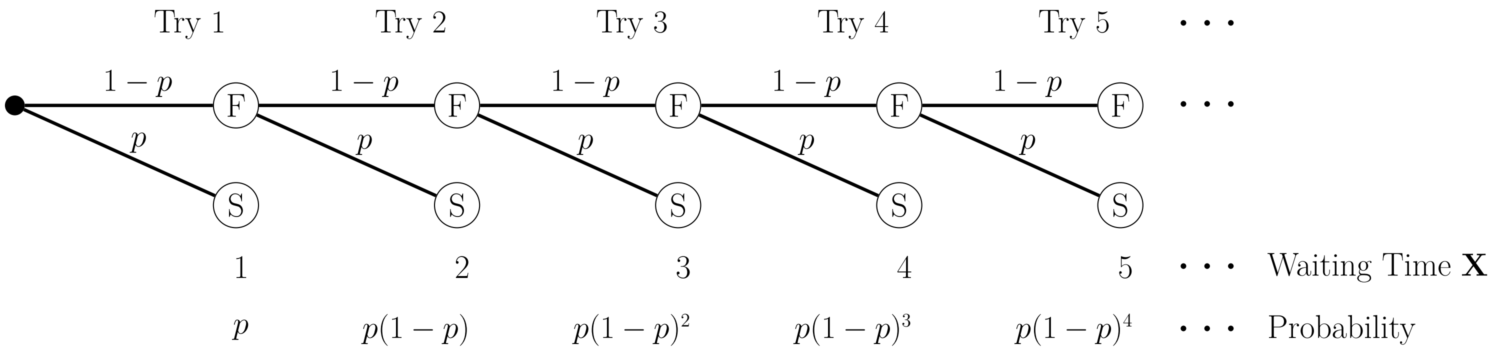
Exponential Random Variable: Waiting Time to Success

Let p be the probability to succeed on a trial.



Exponential Random Variable: Waiting Time to Success

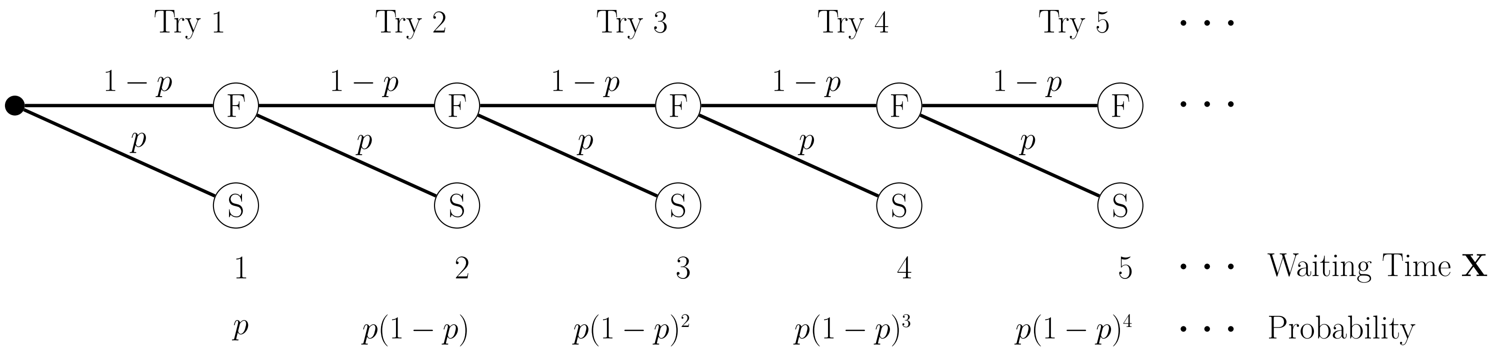
Let p be the probability to succeed on a trial.



$$\mathbb{P}[t \text{ trials}] = \mathbb{P}[\mathrm{F}^{\bullet t-1}\mathrm{S}] = (1-p)^{t-1}p$$

Exponential Random Variable: Waiting Time to Success

Let p be the probability to succeed on a trial.

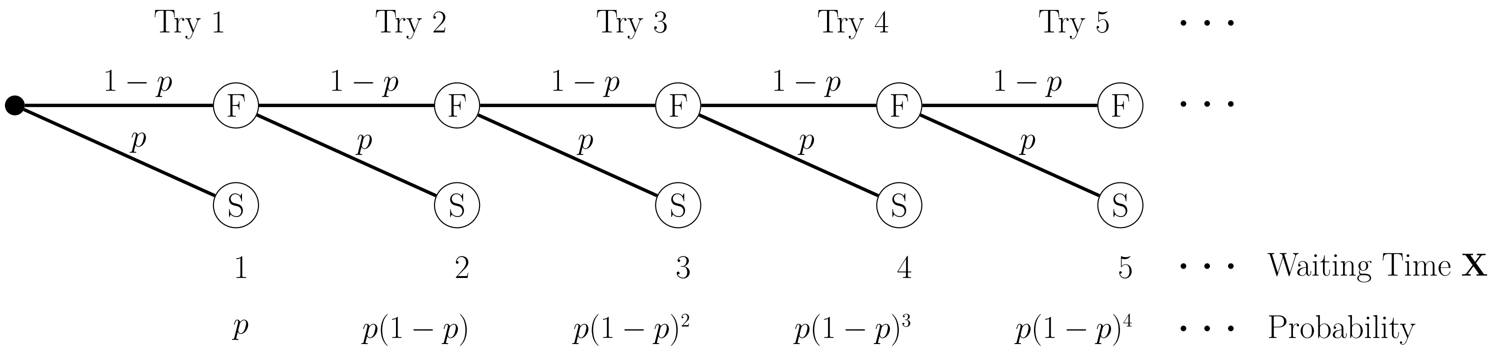


$$\mathbb{P}[t\text{ trials}] = \mathbb{P}[\mathrm{F}^{\bullet t-1}\mathrm{S}] = (1-p)^{t-1}p$$

$$P_{\mathbf{X}}(t) \;=\; (1-p)^{t-1}p \;=\; \underbrace{\frac{p}{1-p}}_{\beta}\times(1-p)^{t}$$

Exponential Random Variable: Waiting Time to Success

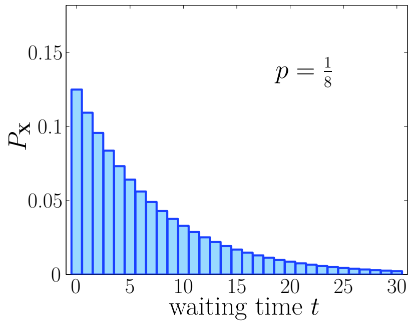
Let p be the probability to succeed on a trial.



$$\mathbb{P}[t \text{ trials}] = \mathbb{P}[\mathrm{F}^{\bullet t-1}\mathrm{S}] = (1-p)^{t-1}p$$

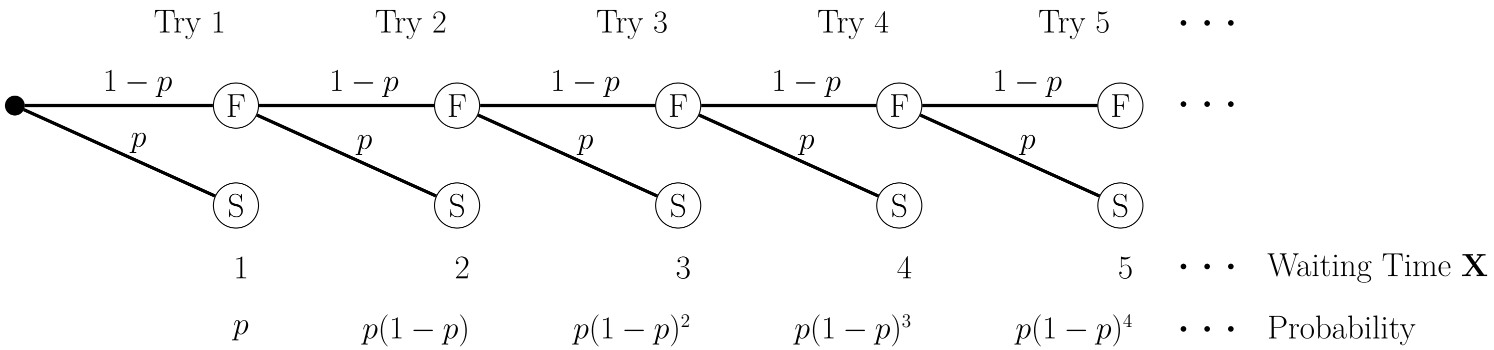
$$P_{\mathrm{X}}(t) \;=\; (1-p)^{t-1}p \;=\; \underbrace{\frac{p}{1-p}}_{\beta}\times(1-p)^t$$

$$=\; \beta(1-p)^t.$$



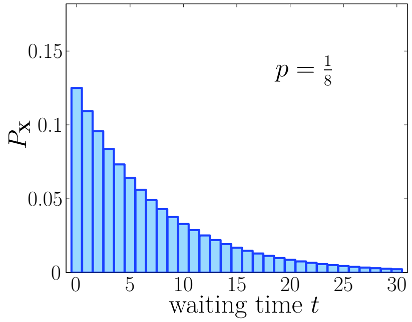
Exponential Random Variable: Waiting Time to Success

Let p be the probability to succeed on a trial.



$$\mathbb{P}[t \text{ trials}] = \mathbb{P}[\text{F}^{\bullet t-1}\text{S}] = (1-p)^{t-1}p$$

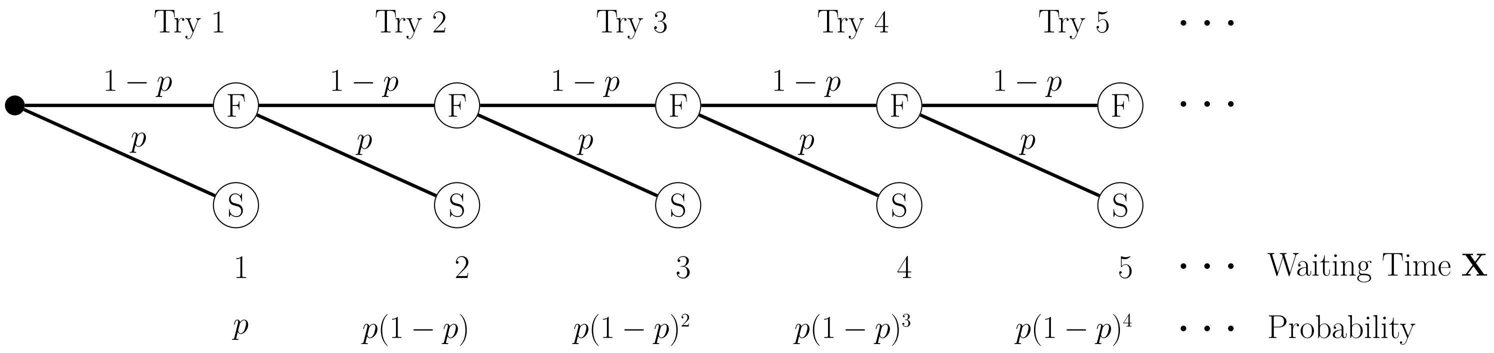
$$\begin{aligned} P_{\mathbf{X}}(t) &= (1-p)^{t-1}p = \underbrace{\frac{p}{1-p}}_{\beta}\times(1-p)^t \\ &= \beta(1-p)^t. \end{aligned}$$



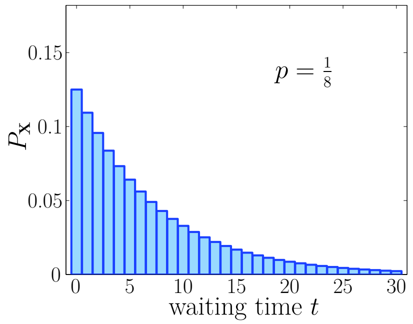
Example: 3 people randomly access the wireless channel. Success only if exactly one is attempting.

Exponential Random Variable: Waiting Time to Success

Let p be the probability to succeed on a trial.



$$\mathbb{P}[t \text{ trials}] = \mathbb{P}[\text{F}^{\bullet t-1}\text{S}] = (1-p)^{t-1}p$$



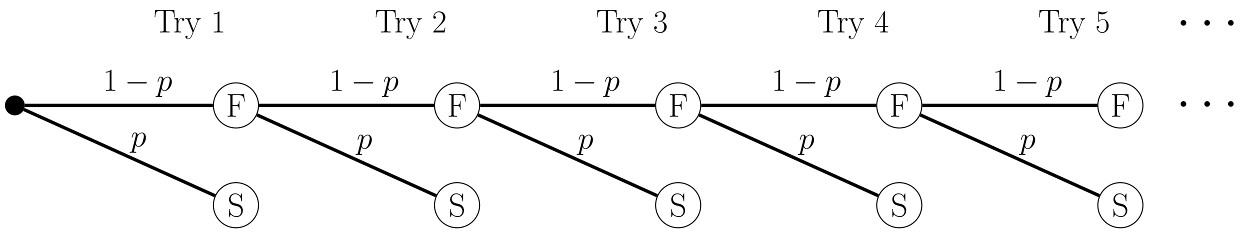
$$\begin{aligned} P_{\text{X}}(t) &= (1-p)^{t-1}p = \underbrace{\frac{p}{1-p}}_{\beta} \times (1-p)^t \\ &= \beta(1-p)^t. \end{aligned}$$

Example: 3 people randomly access the wireless channel. Success only if exactly one is attempting.

Try every timestep

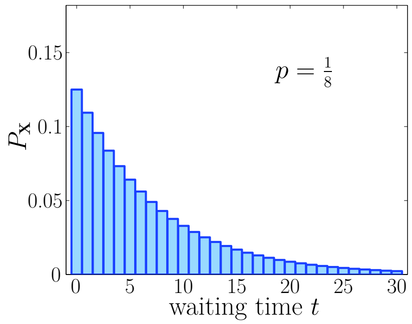
Exponential Random Variable: Waiting Time to Success

Let p be the probability to succeed on a trial.



1	2	3	4	5	⋯	Waiting Time **X**
p	$p(1-p)$	$p(1-p)^2$	$p(1-p)^3$	$p(1-p)^4$	⋯	Probability

$$\mathbb{P}[t \text{ trials}] = \mathbb{P}[\text{F}^{\bullet t-1}\text{S}] = (1-p)^{t-1}p$$



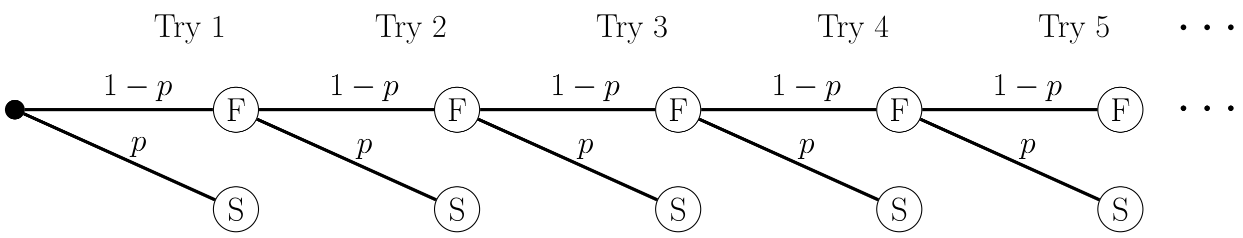
$$\begin{aligned} P_{\text{X}}(t) &= (1-p)^{t-1}p = \underbrace{\frac{p}{1-p}}_{\beta} \times (1-p)^t \\ &= \beta(1-p)^t. \end{aligned}$$

Example: 3 people randomly access the wireless channel. Success only if exactly one is attempting.

Try every timestep → no one succeeds.

Exponential Random Variable: Waiting Time to Success

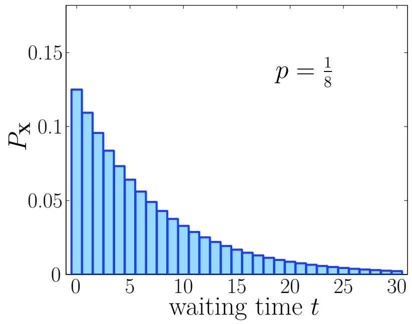
Let p be the probability to succeed on a trial.



	1	2	3	4	5	· · ·	Waiting Time **X**
	p	$p(1-p)$	$p(1-p)^2$	$p(1-p)^3$	$p(1-p)^4$	· · ·	Probability

$$\mathbb{P}[t \text{ trials}] = \mathbb{P}[\mathrm{F}^{\bullet t-1}\mathrm{S}] = (1-p)^{t-1}p$$

$$\begin{aligned} P_{\mathbf{X}}(t) &= (1-p)^{t-1}p = \underbrace{\frac{p}{1-p}}_{\beta}\times(1-p)^{t} \\ &= \beta(1-p)^{t}. \end{aligned}$$

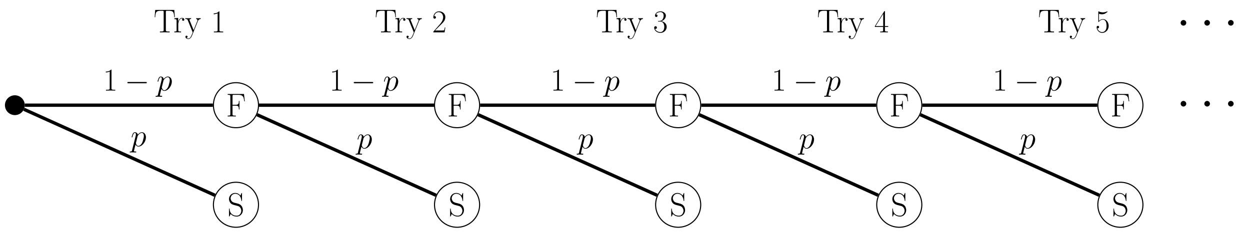


Example: 3 people randomly access the wireless channel. Success only if exactly one is attempting.

Try every timestep → no one succeeds. Everyone tries $\frac{1}{3}$ the time (randomly).

Exponential Random Variable: Waiting Time to Success

Let p be the probability to succeed on a trial.

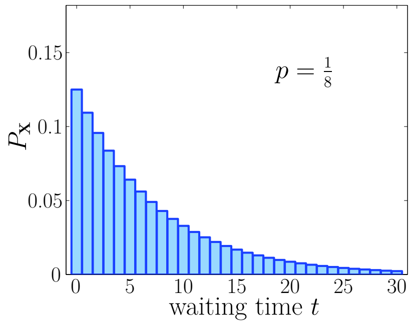


1	2	3	4	5	· · ·	Waiting Time **X**
p	$p(1-p)$	$p(1-p)^2$	$p(1-p)^3$	$p(1-p)^4$	· · ·	Probability

$$\mathbb{P}[t \text{ trials}] = \mathbb{P}[\text{F}^{\bullet t-1}\text{S}] = (1-p)^{t-1}p$$

$$P_{\text{X}}(t) = (1-p)^{t-1}p = \underbrace{\frac{p}{1-p}}_{\beta} \times (1-p)^t$$

$$= \beta(1-p)^t.$$

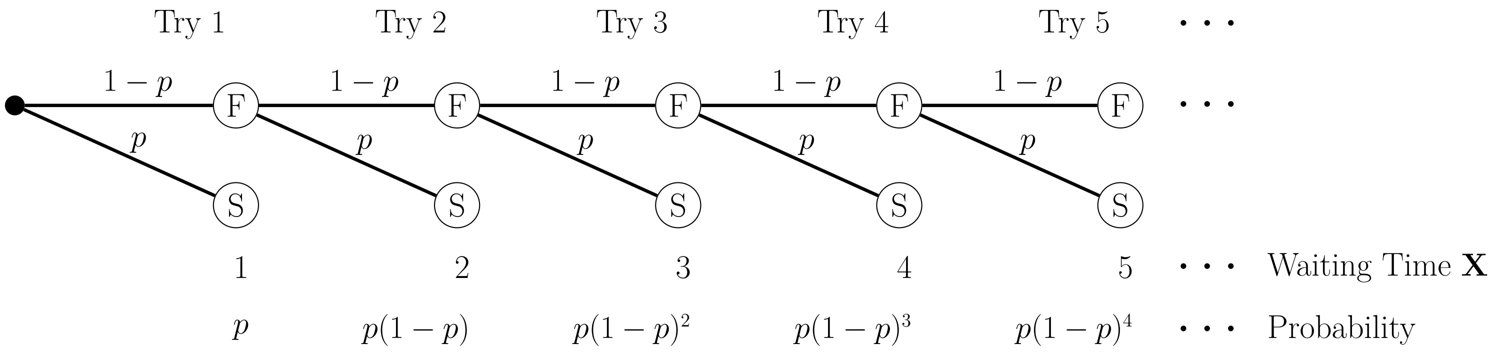


Example: 3 people randomly access the wireless channel. Success only if exactly one is attempting.

Try every timestep → no one succeeds. Everyone tries $\frac{1}{3}$ the time (randomly).
Success probability for *someone* is $\frac{4}{9}$.

Exponential Random Variable: Waiting Time to Success

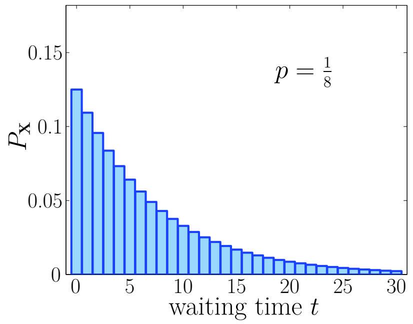
Let p be the probability to succeed on a trial.



$$\mathbb{P}[t \text{ trials}] = \mathbb{P}[\mathrm{F}^{\bullet t-1}\mathrm{S}] = (1-p)^{t-1}p$$

$$P_{\mathrm{X}}(t) = (1-p)^{t-1}p = \underbrace{\frac{p}{1-p}}_{\beta} \times (1-p)^t$$

$$= \beta(1-p)^t.$$

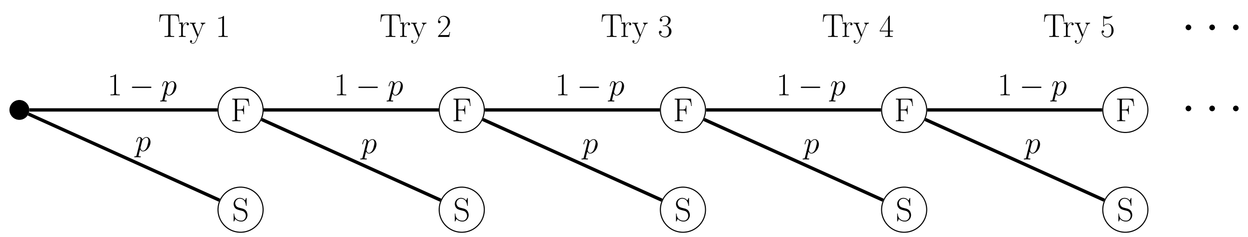


Example: 3 people randomly access the wireless channel. Success only if exactly one is attempting.

Try every timestep → no one succeeds. Everyone tries $\frac{1}{3}$ the time (randomly).
Success probability for *someone* is $\frac{4}{9}$. Success probability for *you* is $\frac{4}{27}$.

Exponential Random Variable: Waiting Time to Success

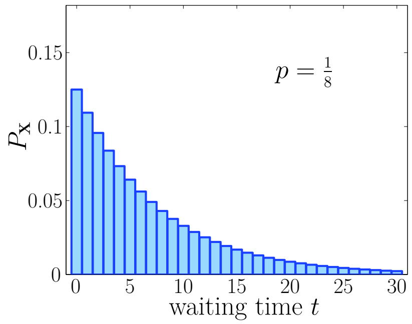
Let p be the probability to succeed on a trial.



	1	2	3	4	5	$\cdots$
Waiting Time **X**	1	2	3	4	5	$\cdots$
Probability	p	$p(1-p)$	$p(1-p)^2$	$p(1-p)^3$	$p(1-p)^4$	$\cdots$

$$\mathbb{P}[t \text{ trials}] = \mathbb{P}[\mathrm{F}^{\bullet t-1}\mathrm{S}] = (1-p)^{t-1}p$$

$$\begin{aligned} P_{\mathbf{X}}(t) &= (1-p)^{t-1}p = \underbrace{\frac{p}{1-p}}_{\beta}\times(1-p)^t \\ &= \beta(1-p)^t. \end{aligned}$$



Example: 3 people randomly access the wireless channel. Success only if exactly one is attempting.

Try every timestep → no one succeeds. Everyone tries $\frac{1}{3}$ the time (randomly).

Success probability for *someone* is $\frac{4}{9}$. Success probability for *you* is $\frac{4}{27}$.

wait, t	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\mathbb{P}$[someone succeeds]	0.444	0.247	0.137	0.076	0.042	0.024	0.013	0.007	0.004	0.002	0.001	$\cdots$
$\mathbb{P}$[you succeed]	0.148	0.126	0.108	0.092	0.078	0.066	0.057	0.048	0.051	0.035	0.030	$\cdots$