QUIZ 3: 110 Minutes

Answer **ALL** questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

You **MUST** show **CORRECT** work to get full credit.

When in doubt, TINKER.

1	2	3	Total
150	25	25	200

1 **Circle one answer per question.** 15 **points for each correct answer.**

(a) Shantelle is estimating how long her weekly grocery shopping will take. She plans to visit four stores, and models the number of minutes she will spend in line at each store using Geo($\frac{1}{2}$), Geo($\frac{1}{3}$), Geo($\frac{1}{6}$), and Geo($\frac{1}{5}$) random variables, respectively. How many minutes should she expect to spend waiting in line?

A $1\frac{1}{5}$

B 10

C 15

~~D~~ 16

E None of the above.

$W = X_1 + X_2 + X_3 + X_4$ where $X_i \sim \text{Geo}(p_i)$

$\Rightarrow \mathbb{E}W = \mathbb{E}X_1 + \mathbb{E}X_2 + \mathbb{E}X_3 + \mathbb{E}X_4$

$= \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} = 2+3+6+5$

$= 16$

(b) Which of the following random quantities can be calculated in terms of a Binomial random variable?

(I) Toss a fair coin 101 times, and determine the probability that you see more heads than tails.
(II) The number of darts thrown until you hit a bull's eye.
(III) The number of students taking this exam that will correctly answer this question.
(IV) Each vertex of a graph is randomly placed into a set A or a set B. The graph has m edges. A "cut-edge" is one that has its two vertices in different sets. The expected number of cut-edges.

A I, II, III

~~B~~ I, III, IV

C III, IV

D II, IV

E All of them

(I) $X \sim \text{Bin}(101, 1/2)$, $\mathbb{P}(X \geq 51)$

(II) $X \sim \text{Geo}(p)$ where p = prob hitting bull's eye in one throw

(III) $X \sim \text{Bin}(n, p)$ where n = # students, p = prob answer correctly

(IV) $\mathbb{E}C$, where $C \sim \text{Bin}(m, \frac{1}{2})$

(c) Consider a graph with 10 nodes $v_1, \ldots, v_{10}$ in which every possible edge (v_i, v_j), with $i \neq j$, is present with probability p, and the edges are independently present[1]. What is the pdf of the number of edges in the graph?

A $\mathbb{P}(\mathbf{E} = k) = (1-p)^{10-k} p^k$

B $\mathbb{P}(\mathbf{E} = k) = (1-p)^{\binom{10}{2}-k} p^k$

C $\mathbb{P}(\mathbf{E} = k) = \frac{k}{\binom{10}{2}}$

D $\mathbb{P}(\mathbf{E} = k) = \frac{\binom{10}{2} p^k}{2^{\binom{10}{2}}}$

~~E~~ None of the above.

$\mathbb{P}(E = k) = \underbrace{\binom{\binom{10}{2}}{k} p^k (1-p)^{\binom{10}{2}-k}}_{\text{probability that exactly } k \text{ edges are present}}$

[1] Informally: whether one edge is present tells me nothing about whether the other edges are present.

(d) Which of the following claims is true?

A If $S \subseteq \mathbb{R}$ is a countable set, then $\overline{S} \subseteq \mathbb{R}$ is uncountable.

B There is an uncountable subset of $\mathbb{Z} \times \mathbb{Z}$.

C The union of uncountably many languages is a language.

~~D~~ Exactly two of the above.

E None of the above.

Since $\mathbb{R} = S \cup \overline{S}$, if both S & $\overline{S}$ were countable, $\mathbb{R}$ would be as well, so $\overline{S}$ is uncountable

$\mathbb{Z} \times \mathbb{Z}$ is countable, so any subset is as well

every language is a subset of Σ^*, so the union is also a subset of Σ^*, and therefore countable

(e) Which of the following strings is *not* in the language

$$\mathcal{L} = \underbrace{\overline{(0^*10^*) \cup (0^*)}}_{\text{strings w/ at most one 1}} \cap \underbrace{\overline{(1^*01^*) \cup (1^*)}}_{\text{strings w/ at most one 0}} \ ?$$

~~A~~ ε

B 0011

C 1010

D 11001

E All of the above are in the language

$\Rightarrow \mathcal{L}$ = strings w/ at least two ones and two zeroes

(f) Let $\mathbf{X}$ be uniformly distributed on $\{5, 6, \ldots, 11\}$. Compute $\mathbb{E}[\mathbf{X}^2 \mid \mathbf{X} \text{ is prime}]$.

A $\frac{476}{3}$

B $\frac{585}{7}$

~~C~~ $\frac{195}{3}$

D $\frac{476}{7}$

E None of the above.

$$\mathbb{E}[X^2 \mid X \text{ is prime}] = \sum_{x=5}^{11} x^2 \, \mathbb{P}(X = x \mid X \text{ is prime})$$

$$= \frac{5^2}{3} + \frac{7^2}{3} + \frac{11^2}{3} = \frac{25 + 49 + 121}{3}$$

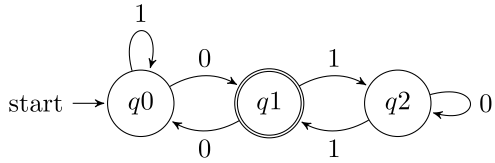
$$= \frac{195}{3}$$

because

$$\mathbb{P}(X = x \mid X \text{ is prime}) = \frac{\mathbb{P}(X = x \text{ and } X \text{ is prime})}{\mathbb{P}(X \text{ is prime})} = \begin{cases} 0 & \text{if } x \text{ composite} \\ \dfrac{\mathbb{P}(X=x)}{\mathbb{P}(X \text{ is prime})} & \text{if } x \text{ prime} \end{cases}$$

$$= \begin{cases} 0 & \text{if } x \text{ composite} \\ \dfrac{1/7}{3/7} = \dfrac{1}{3} & \text{if } x \text{ prime} \end{cases}$$

(g) Consider the simple computing machine



Which of the following strings will it accept?

[A] 1010

[B] 1001

[C] 10101 ✗

[D] 100001

[E] None of the above

(h) Identify the proper relationships between the cardinalities of the sets $\mathbb{Q}$, $\mathbb{Z} \times \mathbb{Q}$, and $\mathbb{Z}$.

[A] $|\mathbb{Z} \times \mathbb{Q}| > |\mathbb{Q}| > |\mathbb{Z}|$

[B] $|\mathbb{Z} \times \mathbb{Q}| > |\mathbb{Q}| = |\mathbb{Z}|$

[C] $|\mathbb{Z} \times \mathbb{Q}| = |\mathbb{Q}| > |\mathbb{Z}|$

[D] $|\mathbb{Z} \times \mathbb{Q}| = |\mathbb{Q}| = |\mathbb{Z}|$ ✗

[E] None of the above

We know $\mathbb{Q}$, $\mathbb{Z}$ are countable. We proved in class (in showing $\mathbb{Q}$ is countable), that $\mathbb{Z} \times \mathbb{N}$ is countable. Same idea shows $\mathbb{Z} \times \mathbb{Q}$ is:

$\mathbb{Z} \backslash \mathbb{Q}$	1	$1/2$	$1/3$	$\cdots$	$7/5$	$\cdots$
0	$(0,1)$	$(0,1/2)$	$(0,1/3)$			
-1	$(-1,1)$	$(-1,1/2)$	$(-1,1/3)$			
2	$(2,1)$	$(2,1/2)$	$(2,1/3)$			
-2	$(-2,1)$	$(-2,1/3)$	$(-2,1/3)$			
$\vdots$						

(i) FOCSbits are the latest cryptographic currency. One FOCSbit can be mined using a randomized algorithm that runs in time $\mathbf{M} \sim \text{Geo}(p)$ for some p. The value of a FOCSbit (in US dollars) is derived from the amount of time it took to mine it; specifically, $\mathbf{V} = \mathbf{M}^2$. Find the expected value of a FOCSbit. *Hint: use total expectation.*

[A] $\frac{1}{p^2}$

[B] $\frac{2}{p^2}$

[C] $\frac{2-p}{p^2}$ ✗

[D] $\frac{1-p^2}{2p}$

[E] None of the above

This is in-chapter example 20.5:

$$\mathbb{E}[M^2] = \mathbb{E}[M^2 \mid \text{fail on first trial}]\,\mathbb{P}[\text{fail on first trial}] + \mathbb{E}[M^2 \mid \text{succeed on first trial}]\,\mathbb{P}[\text{succeed on first trial}]$$

$$= \mathbb{E}[(M'+1)^2](1-p) + 1 \cdot p \quad \text{where } M' \sim \text{Geo}(p)$$

$$= (\mathbb{E}[M^2] + 2\,\mathbb{E}[M] + 1)(1-p) + p$$

$$= (1-p)\,\mathbb{E}[M^2] + \frac{2(1-p)}{p} + 1$$

solve eqn to get $\mathbb{E}M^2$

(j) Which of the following sets is *not* countable?

- $\boxed{\text{A}}$ $\{1, 2, 3, 4\}$ ← finite, so countable
- $\boxed{\text{B}}$ All possible English novels. ← injection exists to Σ^*: $f(\text{novel}) = \text{novel in ASCII}$, so countable
- ~~$\boxed{\text{C}}$~~ All languages (languages as defined in this portion of the course).
- $\boxed{\text{D}}$ All rooted binary trees.
- $\boxed{\text{E}}$ None: all of the above sets are in fact countable.

on HW showed set of all unweighted graphs is countable; this is a subset, so countable

1) bijection exists between this set and infinite binary sequences: list the strings in $\Sigma^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \dots\}$ and take $f(\mathcal{L}) = 0110100\cdots$ where ith entry is one iff the ith string in Σ^* is in $\mathcal{L}$. Set of infinite binary strings is uncountable, so this set is as well

or

2) Cantor's theorem: this set is power set of Σ^*, so has cardinality greater than Σ^*, so is uncountable

see discussion at end of §22.2

2 **Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ be i.i.d. random variables uniformly distributed on $\{1, \ldots, 10\}$. Find the pdf of the random variable $\mathbf{Y} = \min(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$. *Hint: compute* $\mathbb{P}(\mathbf{Y} \geq i)$.**

Essentially in-class exercise in Lecture 18

We use the fact that

$$\mathbb{P}(Y \geq i) = \sum_{j=i}^{10} \mathbb{P}(Y = j) = \mathbb{P}(Y = i) + \mathbb{P}(Y \geq i+1)$$

so pdf can be obtained as

$$\mathbb{P}(Y = i) = \mathbb{P}(Y \geq i) - \mathbb{P}(Y \geq i+1),$$

as in the in-class example in Lecture 18.

Computing,

$$\mathbb{P}(Y \geq i) = \mathbb{P}(X_1 \geq i \text{ and } X_2 \geq i \text{ and } X_3 \geq i)$$

$$= \mathbb{P}(X_1 \geq i)^3 \quad \text{because } X_1, X_2, X_3 \text{ are i.i.d.}$$

$$= \left(\frac{11-i}{10}\right)^3 \quad \text{because } X_1 \text{ is uniform on } [10].$$

So the pdf is

$$\boxed{\mathbb{P}(Y = i) = \left(\frac{11-i}{10}\right)^3 - \left(\frac{10-i}{10}\right)^3 \quad \text{for } i = 1, \ldots, 10}$$

3 **Prove the following generalization of the fact that there are non-regular languages: there are languages that cannot be described using *any* finite length description. Specifically, assume that a finite length description uses only the 255 characters of the ASCII code, but can have arbitrary length. Use a counting argument to prove that there is a language that cannot be described using a finite length description.**

Prf

The set of languages is uncountable: see the bijection constructed between it and the set of infinite length binary strings in the answer to the last MC problem.

On the other hand, the set of finite length descriptions is countable: given a finite length description, convert each of its characters to the ASCII code and concatenate them. This gives an injection from the set of finite length descriptions to Σ^*, which is countable.

Now, for the sake of contradiction, assume that every language has a finite length description. This gives* an injection from the set of languages to the set of finite length descriptions. This is a contradiction, as the cardinality of the former is strictly greater than that of the latter. We conclude that there is at least one language that has no finite length description.

* To be precise about this, you could use the principle of well-ordering to construct a definite injection.

SCRATCH

SCRATCH