

QUIZ 3: 110 Minutes

Answer **ALL** questions.

NO COLLABORATION or electronic devices. Any violations result in an F.

NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

You **MUST** show **CORRECT** work to get full credit.

When in doubt, **TINKER**.

1	2	3	Total
150	25	25	200

1 Circle one answer per question. 15 points for each correct answer.

- (a) Shantelle is estimating how long her weekly grocery shopping will take. She plans to visit four stores, and models the number of minutes she will spend in line at each store using $\text{Geo}(\frac{1}{2})$, $\text{Geo}(\frac{1}{3})$, $\text{Geo}(\frac{1}{6})$, and $\text{Geo}(\frac{1}{5})$ random variables, respectively. How many minutes should she expect to spend waiting in line?

- ☐ A $1\frac{1}{5}$
☐ B 10
☐ C 15
☐ D 16
☐ E None of the above.

- (b) Which of the following random quantities can be calculated in terms of a Binomial random variable?

- (I) Toss a fair coin 101 times, and determine the probability that you see more heads than tails.
(II) The number of darts thrown until you hit a bull's eye.
(III) The number of students taking this exam that will correctly answer this question.
(IV) Each vertex of a graph is randomly placed into a set A or a set B . The graph has m edges. A “cut-edge” is one that has its two vertices in different sets. The expected number of cut-edges.

- ☐ A I, II, III
☐ B I, III, IV
☐ C III, IV
☐ D II, IV
☐ E All of them

- (c) Consider a graph with 10 nodes v_1, \dots, v_{10} in which every possible edge (v_i, v_j) , with $i \neq j$, is present with probability p , and the edges are independently present¹. What is the pdf of the number of edges in the graph?

- ☐ A $\mathbb{P}(\mathbf{E} = k) = (1 - p)^{10-k} p^k$
☐ B $\mathbb{P}(\mathbf{E} = k) = (1 - p)^{\binom{10}{2} - k} p^k$
☐ C $\mathbb{P}(\mathbf{E} = k) = \frac{k}{\binom{10}{2}}$
☐ D $\mathbb{P}(\mathbf{E} = k) = \frac{\binom{10}{2} p^k}{2^{\binom{10}{2}}}$
☐ E None of the above.

¹Informally: whether one edge is present tells me nothing about whether the other edges are present.

(d) Which of the following claims is true?

- ☐ A If $S \subseteq \mathbb{R}$ is a countable set, then $\overline{S} \subseteq \mathbb{R}$ is uncountable.
- ☐ B There is an uncountable subset of $\mathbb{Z} \times \mathbb{Z}$.
- ☐ C The union of uncountably many languages is a language.
- ☐ D Exactly two of the above.
- ☐ E None of the above.

(e) Which of the following strings is *not* in the language

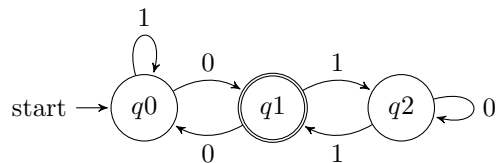
$$\overline{(0^*10^*) \cup (0^*)} \cap \overline{(1^*01^*) \cup (1^*)} \quad ?$$

- ☐ A ε
- ☐ B 0011
- ☐ C 1010
- ☐ D 11001
- ☐ E All of the above are in the language

(f) Let \mathbf{X} be uniformly distributed on $\{5, 6, \dots, 11\}$. Compute $\mathbb{E}[\mathbf{X}^2 \mid \mathbf{X} \text{ is prime}]$.

- ☐ A $\frac{476}{3}$
- ☐ B $\frac{585}{7}$
- ☐ C $\frac{195}{3}$
- ☐ D $\frac{476}{7}$
- ☐ E None of the above.

(g) Consider the simple computing machine



Which of the following strings will it accept?

- ☐ A 1010
- ☐ B 1001
- ☐ C 10101
- ☐ D 100001
- ☐ E None of the above

(h) Identify the proper relationships between the cardinalities of the sets \mathbb{Q} , $\mathbb{Z} \times \mathbb{Q}$, and \mathbb{Z} .

- ☐ A $|\mathbb{Z} \times \mathbb{Q}| > |\mathbb{Q}| > |\mathbb{Z}|$
- ☐ B $|\mathbb{Z} \times \mathbb{Q}| > |\mathbb{Q}| = |\mathbb{Z}|$
- ☐ C $|\mathbb{Z} \times \mathbb{Q}| = |\mathbb{Q}| > |\mathbb{Z}|$
- ☐ D $|\mathbb{Z} \times \mathbb{Q}| = |\mathbb{Q}| = |\mathbb{Z}|$
- ☐ E None of the above

(i) FOCSbits are the latest cryptographic currency. One FOCSbit can be mined using a randomized algorithm that runs in time $\mathbf{M} \sim \text{Geo}(p)$ for some p . The value of a FOCSbit (in US dollars) is derived from the amount of time it took to mine it; specifically, $\mathbf{V} = \mathbf{M}^2$. Find the expected value of a FOCSbit. *Hint: use total expectation.*

- ☐ A $\frac{1}{p^2}$
- ☐ B $\frac{2}{p^2}$
- ☐ C $\frac{2-p}{p^2}$
- ☐ D $\frac{1-p^2}{2p}$
- ☐ E None of the above

(j) Which of the following sets is *not* countable?

☐ A $\{1, 2, 3, 4\}$

☐ B All possible English novels.

☐ C All languages (languages as defined in this portion of the course).

☐ D All rooted binary trees.

☐ E None: all of the above sets are in fact countable.

- 2** Let X_1, X_2, X_3 be i.i.d. random variables uniformly distributed on $\{1, \dots, 10\}$. Find the pdf of the random variable $Y = \min(X_1, X_2, X_3)$. *Hint: compute $\mathbb{P}(Y \geq i)$.*

- 3** Prove the following generalization of the fact that there are non-regular languages: there are languages that cannot be described using *any* finite length description. Specifically, assume that a finite length description uses only the 255 characters of the ASCII code, but can have arbitrary length. Use a counting argument to prove that there is a language that cannot be described using a finite length description.

SCRATCH

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