# QUIZ 2: 110 Minutes

Answer **ALL** questions.
**NO COLLABORATION or electronic devices. Any violations result in an F.**
**NO questions** allowed during the test. Interpret and do the best you can.

# GOOD LUCK!

You **MUST** show **CORRECT** work to get full credit.

**When in doubt, TINKER.**

| 1 | 2 | 3 | Total |
|---|---|---|---|
| | | | |
| 150 | 25 | 25 | 200 |

## 1 Circle one answer per question. 15 points for each correct answer.

(a) Let $C_n$ be the cycle graph on $n$ vertices. What is $\chi(C_n)$?

A 2

B 3

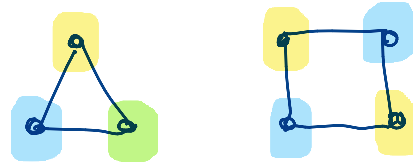
C 4

D None of the above.

E Not enough information.

either is fine

$$\chi(C_n) = \begin{cases} 2, & 2 \mid n \quad (n>1), \text{ e.g.} \\ 3, & 2 \nmid n \end{cases}$$



(b) How many functions $f : \{1, \ldots, 5\} \to \{1, \ldots, 10\}$ are *strictly* increasing?

A $\binom{14}{4}$

B $\binom{14}{5}$

C $\binom{15}{5}$

D $\binom{15}{9}$

E None of the above.

$f(1) < f(2) < \cdots < f(5)$ correspond 1-1 with selecting 5 elements from $[10]$, so $\boxed{\binom{10}{5}}$

(c) A social network has seven people Ⓐ, Ⓑ, …, Ⓖ. Summing up the number of friends of each person in the network gives 26. How many *different* such social networks are there?

A $7 \cdot 13$

B $\frac{13!}{6!}$

C $2^{\binom{7}{2}/26}$

D $\binom{\binom{7}{2}}{26}$

E None of the above.

There are $\binom{7}{2}$ possible edges in a graph w/ 7 vertices. From handshaking thm, $|E| = \frac{\sum d_i}{2} = 13$ so choosing 13 edges from $\binom{7}{2}$ possible

$\Rightarrow \boxed{\binom{\binom{7}{2}}{13}}$

(d) A vase contains $r$ red balls and $b$ blue balls. A ball is chosen at random from the vase, its color is noted, and it is returned to the vase together with $d$ more balls of the same color. This is repeated indefinitely. What is the probability that the second ball is blue?

A $\frac{b}{b+r+d}$

B $\frac{b}{b+r}$

C $\frac{b+d}{b+r+d}$

D $\frac{b+d}{b+r}$

E None of the above.

$$\mathbb{P}(\text{second ball is blue}) = \mathbb{P}(b|b)\mathbb{P}(b) + \mathbb{P}(b|r)\mathbb{P}(r)$$

$$= \frac{b}{b+r} \cdot \frac{b+d}{b+d+r} + \frac{r}{b+r}\frac{b}{b+r+d}$$

$$= \frac{b^2 + bd + rb}{(b+r)(b+d+r)} = \frac{b(b+d+r)}{b+r\,(b+d+r)} = \frac{b}{b+r}$$

(e) You randomly choose two gloves, without replacement, from a drawer with seven pairs of gloves. What is the probability that the gloves are a matching pair?

A 1/14

~~B~~ 1/13

~~C~~ $7/\binom{14}{2}$

D $14/\binom{14}{2}$

E None of the above

either is fine

$7/\binom{14}{2} = \frac{1}{13}$

$\mathbb{P}(\text{match}) = \mathbb{P}(\text{second glove matches first})$

$= \frac{1}{13}$

(f) The first round of a karate tournament pairs the fighters into groups of two. If there are 16 fighters, how many ways are there of forming the first round?

A $\frac{16!}{8!2}$

B $\binom{16}{2}\binom{14}{2}\cdots\binom{2}{2}$

C $\binom{21}{7}$

D $\binom{21}{16}$

~~E~~ None of the above.

each round can be obtained by permuting $1, \ldots, 16$ and splitting into eight pairs. but the order of those eight pairs don't matter, and the order of the elements in each of the eight pairs don't matter, so you get $\boxed{\frac{16!}{2^8 8!}}$

(g) Of 50 students in Ethics, Statistics, and Circuits and Systems, the number failing each combination of courses is shown. How many students passed all the courses?

| E | S | C | ES | EC | SC | ESC |
|---|---|---|---|---|---|---|
| 10 | 5 | 5 | 2 | 4 | 2 | 1 |

A 28

B 31

C 35

~~D~~ 37

E None of the above

$|E^c \cap S^c \cap C^c| = 50 - |E \cup S \cup C|$

$= 50 - (|E| + |S| + |C| - |ES| - |EC| - |SC| + |ESC|)$

$= 50 - (10 + 5 + 5 - 2 - 4 - 2 + 1)$

$= 50 - (13) = 37$

(h) A bag has 4 coins: two 2-headed coins, a 2-tailed coin, and a regular fair coin. Randomly pick a coin and place it on the table. You see a heads facing up. What is the probability that the side facing down is heads?

A 1/4

B 1/2

C 3/4

~~D~~ 4/5

E None of the above

$\mathbb{P}(\text{two-headed} \mid \text{head}) = \frac{\mathbb{P}(\text{two-headed} \cap \text{head})}{\mathbb{P}(\text{head})}$

$= \frac{\mathbb{P}(\text{two-headed})}{[\mathbb{P}(\text{head} \mid \text{two-headed})\mathbb{P}(\text{two-headed}) + \mathbb{P}(\text{head} \mid \text{two-tailed})\mathbb{P}(\text{two-tailed}) + \mathbb{P}(\text{head} \mid \text{fair})\mathbb{P}(\text{fair})]}$

$= \frac{2/4}{\frac{2}{4}\cdot 1 + 0 \cdot \frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2}} = \frac{4}{5}$

(i) Independently generate a length five binary string $b_0 b_1 \cdots b_4$ with $\mathbb{P}[b_i = 0] = 1/2$ for each $i$. What is the probability that the string is sorted from low to high?

A 5/32

~~B~~ 3/16

C 1/8

D 1/2

E None of the above

$\mathbb{P}(\text{string sorted low to high}) = \mathbb{P}(\{00000, 00001, 00011, 00111, 01111, 11111\})$

$= \frac{6}{2^5} = \frac{6}{32} = \frac{3}{16}$

(j) A bucket contains twenty dates. Fifteen of these dates have had their seed removed. A sly dog eats five dates, selected at random. Subsequently, a date is randomly selected from the remaining dates. What is the probability that this date contains a seed?

A $\sum_{k=0}^{5} \frac{\binom{5}{k}}{\binom{20}{5}} \frac{5-k}{15}$

B $\sum_{k=0}^{5} \frac{\binom{5}{k}}{\binom{20}{k}} \frac{5-k}{15}$

C $\sum_{k=0}^{5} \frac{\binom{5}{k}}{\binom{20}{5}} \frac{5-k}{20}$

D $\sum_{k=0}^{5} \frac{\binom{5}{k}}{\binom{20}{k}} \frac{5-k}{20}$

~~E~~ None of the above

$$\mathbb{P}\{\text{date 6 has a seed}\} = \sum_{k=0}^{5} \mathbb{P}\{\text{date 6 has a seed} \mid \text{dog ate k seeds}\} \times \mathbb{P}\{\text{dog ate k seeds}\}$$

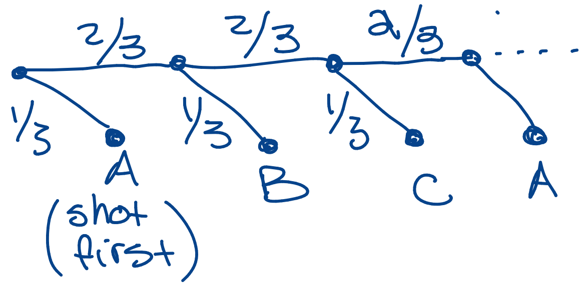
$$= \sum_{k=0}^{5} \frac{5-k}{20-5} \cdot \frac{\binom{5}{k}\binom{15}{5-k}}{\binom{20}{5}}$$

$$= \boxed{\sum_{k=0}^{5} \frac{5-k}{15} \frac{\binom{5}{k}\binom{15}{5-k}}{\binom{20}{5}}}$$

btw, this simplifies to $\frac{1}{4}$

**2** Three monkeys $A, B, C$ have a 6-shooter pistol loaded with 2 bullets. Starting with $A$, each spins the bullet-wheel to a chamber uniformly randomly selected from the six chambers and shoots their foot. They repeat this process cyclically in the order A, B, then C, until there are no bullets left. Compute probabilities $p_A$, $p_B$, $p_C$ for each monkey to be the first shot.

Draw a probability tree:




See that

$$p_A = \mathbb{P}(A \text{ shot first}) = \mathbb{P}(\text{gun first fires at shot } 1, 4, 7, \dots)$$
$$= \mathbb{P}(\text{gun first fires at time in } \{k : k = 3t+1 \text{ for } t \geq 0\})$$
$$= \sum_{t=0}^{\infty} \left(\frac{2}{3}\right)^{3t} \left(\frac{1}{3}\right) = \frac{1}{3} \sum_{t=0}^{\infty} \left(\frac{8}{27}\right)^t = \frac{1}{3}\left(\frac{1}{1 - 8/27}\right)$$
$$= \frac{1}{3} \cdot \frac{27}{19} = \frac{9}{19}$$

$$p_B = \mathbb{P}(B \text{ shot first}) = \mathbb{P}(\text{gun first fires at time } k = 3t+2 \text{ for } t \geq 0)$$
$$= \sum_{t=0}^{\infty} \left(\frac{2}{3}\right)^{3t+1} \left(\frac{1}{3}\right) = \frac{2}{9} \sum_{t=0}^{\infty} \left(\frac{8}{27}\right)^t = \frac{2}{9} \cdot \frac{27}{19} = \frac{6}{19}$$

$$p_C = \mathbb{P}(\text{gun first fires at time } k = 3t \text{ for } t \geq 1)$$
$$= \sum_{t=1}^{\infty} \left(\frac{2}{3}\right)^{3t-1} \left(\frac{1}{3}\right) = \frac{1}{2} \sum_{t=1}^{\infty} \left(\frac{8}{27}\right)^t = \frac{1}{2}\left[\frac{27}{19} - 1\right]$$
$$= \frac{1}{2}\left[\frac{8}{19}\right] = \frac{4}{19}$$

of course you could use $p_C = 1 - p_A - p_B$

$$\boxed{p_A = \frac{9}{19}, \quad p_B = \frac{6}{19}, \quad p_C = \frac{4}{19}}$$

**3** **Label the vertices of $K_{15}$ with the integers $1, \ldots, 15$, and construct a graph $G$ by adding a vertex $16$ and adding edges connecting it to vertices $5, 10$, and $15$. How many perfect matchings are there on $G$?**

A perfect matching on $G$ must match vertex 16 with vertex 5, 10, or 15. Notice that matching 16 to any of these three vertices leaves a copy of $K_{14}$ that must be matched, so the total number of perfect matchings on $G$ is three times the number of perfect matchings on $K_{14}$.

To count the perfect matching on $K_{14}$, use the following counting argument:

- each matching can be obtained by permuting $1, \ldots, 14$ and partitioning this list into 7 pairs
- this mapping from permutations of $[14]$ to matchings has multiplicity $2^7 7!$

So we see there are

$$\boxed{\frac{3 \cdot 14!}{2^7 7!}}$$

perfect matchings on $G$.

**SCRATCH**

# SCRATCH