QUIZ 1: 110 Minutes

Answer **ALL** questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

You **MUST** show **CORRECT** work to get full credit.

When in doubt, TINKER.

1	2	3	Total
150	25	25	200

1 Circle one answer per question. 15 points for each correct answer.

(a) Let $S(x) =$ "x is a science major" and $F(x) =$ "x is a fan of Cosmos". Which of the following translates to "all fans of Cosmos are science majors"?

$\boxed{\text{A}}$ $\neg(\exists x : F(x) \wedge (\neg S(x)))$ $\overset{\text{eqv}}{\equiv}$ $\forall x: \neg F(x) \vee S(x)$ $\overset{\text{eqv}}{\equiv}$ $\forall x: F(x) \to S(x)$

all fans of Cosmos are science majors

$\boxed{\text{B}}$ $\forall x : (\neg F(x)) \vee (\neg S(x))$

$\boxed{\text{C}}$ $\forall x : (\neg S(x)) \to F(x)$

$\boxed{\text{D}}$ All of the above.

$\boxed{\text{E}}$ None of the above.

(b) Let p be prime. Which most accurately describes the number $\sqrt{p}$?

$\boxed{\text{A}}$ A natural number.

$\boxed{\text{B}}$ A rational number.

$\boxed{\text{C}}$ An irrational number. see recitation 4.22(b)

$\boxed{\text{D}}$ It depends on the value of p.

$\boxed{\text{E}}$ None of the above.

(c) If x and y denote integers, which of the following propositions are true?

(1) $\forall x : (\exists y : 2x - y = 0)$ ← take $y=2x$

(2) $\exists y : (\forall x : 2x - y = 0)$

(3) $\exists y : (\forall x : xy = 0)$ ← take $y=0$

$\boxed{\text{A}}$ (1) and (2).

$\boxed{\text{B}}$ (2) and (3).

$\boxed{\text{C}}$ (1) and (3).

$\boxed{\text{D}}$ All of the above.

$\boxed{\text{E}}$ None of the above.

(d) Let $S = \{(a, b) : a, b \in \mathbb{Q} \text{ and } a < b\}$. Give the negation of the following claim:

$$\forall (a, b) \in S : (\exists t \in \mathbb{Q} : (a < t) \wedge (t < b)).$$

negates to

$\exists (a,b) \in S : \neg(\exists t \in \mathbb{Q} : (a<t) \wedge (t<b))$

$\overset{\text{eqv}}{\equiv}$

$\exists (a,b) \in S : \forall t \in \mathbb{Q} : (a \geq t) \vee (t \geq b)$

$\boxed{\text{A}}$ $\exists (a, b) \in S : (\exists t \in \mathbb{Q} : (a < t) \wedge (t < b))$

$\boxed{\text{B}}$ $\forall (a, b) \in S : (\forall t \in \mathbb{Q} : (a \geq t) \vee (t \geq b))$

$\boxed{\text{C}}$ $\exists (a, b) \in S : (\exists t \in \mathbb{Q} : (a \geq t) \vee (t \geq b))$

$\boxed{\text{D}}$ $\forall (a, b) \in S : (\exists t \in \mathbb{Q} : (a < t) \wedge (t < b))$

$\boxed{\text{E}}$ None of the above.

(e) How many rows in the truth table of $(p \to q) \to (r \wedge s)$ are there where it is the case that this proposition is true?

this proposition fails only if $p \to q = T$, which happens when

p	q
T	T
F	T
F	F

and $r \wedge s = F$, which happens when

r	s
T	F
F	T
F	F

these situation occur on $3 \times 3 = 9$ lines of the truth table, so the remaining $2^4 - 9 = 7$ lines are true

- [A] 6
- [B] 7
- [C] 8
- [D] 9
- [E] 16

(f) Which pair is the (converse, contrapositive) of $(\forall x : (P(x) \wedge Q(x))) \to R(x)$?

$p \to q$ has converse $q \to p$ and contrapositive $\neg q \to \neg p$

- [A] $(\exists x : (\neg P(x)) \vee (\neg Q(x))) \to \neg R(x), \quad \neg R(x) \to (\exists x : (\neg P(x)) \vee (\neg Q(x)))$
- [B] $R(x) \to (\forall x : P(x) \wedge Q(x))\,, \quad \neg R(x) \to (\exists x : (\neg P(x)) \vee (\neg Q(x)))$
- [C] $\neg R(x) \to (\exists x : (\neg P(x)) \vee (\neg Q(x)))\,, \quad R(x) \to (\forall x : P(x) \wedge Q(x))$
- [D] $R(x) \to (\forall x : P(x) \wedge Q(x))\,, \quad (\exists x : (\neg P(x)) \vee (\neg Q(x))) \to \neg R(x)$
- [E] None of the above.

(g) If the proposition "If a person ever $\overbrace{\text{wears a green hat}}^{p}$ then they will $\overbrace{\text{live long or be unlucky in love}}^{q}$" is true, under which condition can we conclude that a person has never worn a green hat?

$\neg q \to \neg p$ where $\neg q$ = die young and be lucky in love

- [A] They lived long and were unlucky in love.
- [B] They died young or were lucky in love.
- [C] They died young and were lucky in love.
- [D] All of the above.
- [E] None of the above.

(h) Which type of proof is the most appropriate for proving that $11^n - 6$ is divisible by 5 if n is a natural number?

- [A] Contradiction.
- [B] Contraposition.
- [C] Direct.
- [D] Leaping Induction.
- [E] Weak Induction. this is a for-all claim

(i) You observe the following hand of cards:

$$\text{S 5 P T 3 4.}$$

Each card has a number on one side and a letter on the other. Consider the following rule: if a card has P on it, then the other side must be a 5. To verify that the rule has not been broken in the hand you have seen, which is the fewest number of cards that you must turn over?

the rule has the form "P on one side" → "5 on the other" so is violated only if there are cards that have P on one side and a number other than 5 on the other. The three circled cards could meet this criterion, so need to be checked.

- [A] 1
- [B] 2
- [C] 3
- [D] 4
- [E] 6

(j) Let a, b be natural numbers. Which of the following propositions is most appropriately proven using direct proof?

- [A] There is an even number greater than ab. ← either $ab+1$ or $ab+2$ is even
- [B] $(a+b)^n \geq a^n + b^n$, when n is a natural number. ← induction
- [C] There is a prime number greater than ab. ← contradiction
- [D] All of the above are equally appropriately proven using direct proof.
- [E] Direct proof is inappropriate for all the above propositions.

2 **Prove that $2^{1/p}$, the p-th root of 2, is irrational for any integer $p > 2$. You may use Fermat's Last Theorem, which states that if a, b, c, n are natural numbers with $n > 2$, then $a^n + b^n \neq c^n$.**

Prf

We construct a contradiction. Assume $2^{1/p}$ is rational, so there exist $c, a \in \mathbb{N}$ so that

$$2^{1/p} = \frac{c}{a}.$$

Then $2 = \dfrac{c^p}{a^p}$, so $2a^p = c^p$. Let $b = a$, then we have found $a, b, p \in \mathbb{N}$ with $p > 2$ satisfying

$$a^p + b^p = c^p.$$

This contradicts Fermat's Last Theorem, so we conclude that $2^{1/p}$ is irrational.

$\blacksquare$

3 **Prove that if n and q are natural numbers then there exist unique integers d and r satisfying $n = dq + r$, with $d \in \{0\} \cup \mathbb{N}$ and $0 \leq r < q$.**

Prf

We use a direct proof. There are two cases:

(i) n is a multiple of q

(ii) n is not divisible by q.

In the first case, $r=0$ by definition and $d=\frac{n}{q}$ is unique by the rules of arithmetic.

In the second case, note that the set of multiples of q that are larger than n is nonempty (it contains nq, for instance). By the well ordering principle it contains a minimal element we denote by $m \cdot q$, where $m \in \mathbb{N}$. Take $d = m-1$ and $r = n - dq$. By the minimality of m,

$$dq < n < mq$$

and subtracting out dq from all quantities,

$$0 < r < q.$$

Thus we have found integers d and r satisfying $n = dq + r$ with $d \in \mathbb{N}$ and $0 < r < q$.

To show uniqueness in this case, assume there exist $d' \in \{0\} \cup \mathbb{N}$ and $0 \leq r' < q$ such that

$$n = d'q + r' \quad \text{with } d' \neq d \text{ or } r' \neq r$$

CONT'D next page

Notice that because n is not a multiple of q, the remainder r' satisfies $0 < r' < q$.

Since $n = dq + r$ and $n = d'q + r'$, in fact

(*) $(d - d')q + r - r' = 0.$

We must consider the following cases:

(i) $d = d'$ and $r \neq r'$. In this case, equation (*) states $r = r'$. This is a contradiction.

(ii) $d \neq d'$ and $r = r'$. In this case, equation (*) states $d = d'$. This is a contradiction.

(iii) $d \neq d'$ and $r \neq r'$. In this case, we note that

$$|(d-d')q| \geq |d-d'|q \geq q,$$

and $r - r' < q$. This contradicts

$$(d-d')q + r - r' = 0.$$

In all cases we see that assuming d and r are not unique leads to a contradiction, therefore d and r are unique.

$\blacksquare$

SCRATCH