MIDTERM: 120 Minutes

Sample Soln

NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.
You **MUST** show **CORRECT** work, **even on multiple choice questions**, to get credit.

GOOD LUCK!

1	2	3	4	5	6	Total
100	**20**	**20**	**20**	**20**	**20**	**200**

1 Circle one answer per question. 10 points for each correct answer.

(a) The moon goes through its phases periodically in the following order: new, waxing crescent, first quarter, waxing gibbous, full, waning gibbous, third quarter, and waning crescent, then new again. Today the moon is in its waning crescent phase. What phase will the moon be in 10^{21} phases from now?

0 1 2 3 4 5 6 7

- [A] new.
- [B] waxing crescent.
- [C] first quarter.
- [D] full.
- [E] None of the above.

waning crescent

$(7 + 10^{21}) \bmod 8 =$

$(7 + 2^{21}) \bmod 8 =$

$(7 + 2^{3 \cdot 7}) \bmod 8 = 7$

(b) How many natural numbers less than 21 are coprime with 21?

- [A] 5.
- [B] 9.
- [C] 10.
- [D] 12.
- [E] None of the above.

means can't share divisors (3 & 7)

1 2 ~~3~~ 4 5 ~~6~~ ~~7~~ 8 ~~9~~ 10 11 ~~12~~ 13 ~~14~~ ~~15~~ 16 17 ~~18~~ 19 20

$\Rightarrow$ 12 numbers

(c) Let A be the adjacency matrix of C_{2n}. Which of the following are true of A?

- [A] $4 \mid \left(\sum_{i=1}^{2n} \sum_{j=1}^{2n} A_{ij}\right)$. $\leftarrow$ true: the sum is $2|E| = 2 \cdot (2n) = 4n$
- [B] $\sum_{i=1}^{n} A_{ii} = 0$. $\leftarrow$ true: no vertices are connected to self
- [C] Each row of A sums to 2. $\leftarrow$ true: each vertex has degree 2
- [D] All three of the above options.
- [E] Two of the above options.

(d) The negation of "The reaction to every action is equal and opposite" is:

- [A] "There are actions whose reactions are not equal and not opposite".
- [B] "There are actions whose reactions are either not equal or not opposite".
- [C] "For every action, the reaction is not equal and not opposite".
- [D] "For every action, the reaction is either not equal or not opposite".
- [E] None of the above.

$\neg(\forall a: E(a, r(a)) \wedge O(a, r(a)))$

$\overset{equ}{\equiv} \exists a: \neg E(a, r(a)) \vee \neg O(a, r(a))$

(e) If $T_0 = 1$, $T_1 = 2$, and $T_{n+2} = T_{n+1} + 5T_n$, what is the value of T_4?

- [A] 12
- [B] 17
- [C] 38
- [D] 52
- [E] None of the above

$T_n = T_{n-1} + 5T_{n-2}$

n	0	1	2	3	4
T_n	1	2	7	17	52

(f) What is $3^{2015} \bmod 7$?

A 2

B 3

C 5

D 6

E None of the above

$3^{2015} \equiv 3^{2\cdot 1007+1} \equiv 9^{1007}\cdot 3$

$\equiv 2^{1007}\cdot 3 \equiv 2^{3\cdot 335+2}\cdot 3$

$\equiv 8^{335}\cdot 4\cdot 3 \equiv 1^{335}\cdot 12$

$\equiv 12 \bmod 7 = 5$

(g) Consider the degree sequence $[6, 6, 5, 4, 3, 3, 1]$. Which of the following is true?

A This degree sequence is not graphical.

B This sequence is graphical, and such a graph is disconnected.

C This sequence is graphical, can be realized with a planar graph, and such a graph has 9 faces.

D This sequence is graphical, and such a graph is a tree.

E None of the above.

both 6 degree vertices need to be connected to the degree 1 node, not possible

(h) Which claim below is true?

A $f \in o(g) \to f \in O(g)$.

B $f \in \Theta(g) \to g \in \Theta(f)$.

C $f \in \omega(g) \to g \in O(f)$.

D None of these claims are true.

E All of these claims are true.

Interpret the relations:

"$f < g$" $\to$ "$f \leq g$"

"$f = g$" $\to$ "$g = f$"

"$f > g$" $\to$ "$g \leq f$"

(i) Which of the following asymptotic relationships is correct?

A $(n+1)^{n+1} \in O(n^n)$.

B $(n+1)^{n+1} \in \omega(n^n)$.

C $(n+1)^{n+1} \in o(n^n)$.

D $(n+1)^{n+1} \in \Theta(n^n)$.

E None of the above.

$$\frac{(n+1)^{n+1}}{n^n} = (n+1)\frac{(n+1)^n}{n^n} \geq n+1$$

so $\displaystyle\lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} = \infty$

(j) How many non-isomorphic $\underbrace{\text{connected acyclic graphs}}_{\text{trees}}$ exist that have four vertices?

A 1.

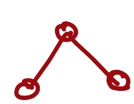
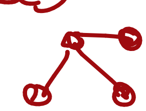
B 2.

C 3.

D 4.

E More than 4.

Any tree on four vertices was constructed by adding a leaf to a tree on three vertices.

The only tree on three vertices is . So the two nonisomorphic trees are  L_4 and  S_4

(Alternatively, enumerate the trees)

2 **Recall the equation for integration by parts, $\int f\,dg = fg - \int g\,df$.The formula for summation by parts is**

$$\sum_{i=m}^{n} f_i(g_{i+1} - g_i) = (f_n g_{n+1} - f_m g_m) - \sum_{i=m+1}^{n} g_i(f_i - f_{i-1}).$$

Choose appropriate sequences f_i and g_i and use summation by parts to show that

$$\sum_{i=0}^{n} i2^i = (n-1)2^{n+1} + 2.$$

Take $f_i = i$ and $g_i = 2^i$, then summation by parts gives

$$\sum_{i=0}^{n} i(2^{i+1} - 2^i) = (n2^{n+1} - 0 \cdot 2^0) - \sum_{i=1}^{n} 2^i(i-(i-1)),$$

or equivalently,

$$2\left(\sum_{i=0}^{n} i2^i\right) - \sum_{i=0}^{n} i2^i = n2^{n+1} - \sum_{i=1}^{n} 2^i,$$

or

$$\sum_{i=0}^{n} i2^i = n2^{n+1} - \left(\sum_{i=0}^{n} 2^i - 1\right)$$

$$= n2^{n+1} - (2^{n+1} - 1 - 1)$$

$$= (n-1)2^{n+1} + 2$$

as claimed.

3 What is the remainder when $6^n + 7^n$ is divided by 8?

We make tables to guess $6^n \bmod 8$ and $7^n \bmod 8$

n	0	1	2	3
6^n	1	6	36	216
$6^n \bmod 8$	1	6	4	0

We prove $6^n \bmod 8 = 0$ when $n \geq 3$ via induction:

<u>Base case</u> $n=3$. $6^n = 216 \equiv 0 \bmod 8$

<u>Inductive step</u> Assume $6^n \bmod 8 = 0$, then

$$6^{n+1} \equiv (6^n \bmod 8) \cdot (6 \bmod 8) \bmod 8$$
$$= 0$$

Therefore $6^n \bmod 8 = \begin{cases} 1, & n=0 \\ 6, & n=1 \\ 4, & n=2 \\ 0 & n \geq 3 \end{cases}$

Likewise

n	0	1	2	3	4
7^n	1	7	49		
$7^n \bmod 8$	1	7	1	7	1

We prove $7^n \bmod 8 = \begin{cases} 1, & 2 \mid n \\ 7, & n \text{ odd} \end{cases}$

via induction:

$\longrightarrow$ cont'd

Base case $n=0$, then $7^0 \equiv 1 \mod 8$

Inductive step Assume $7^n \mod 8 = \begin{cases} 1, & 2|n \\ 7, & n \text{ odd.} \end{cases}$

There are two cases.

Case 1 Assume $2|n$. Then $n+1$ is odd and

$$7^{n+1} \equiv 7^n \cdot 7 \equiv 1 \cdot 7 \mod 8 = 7$$

Case 2 Assume n is odd. Then $n+1$ is even and

$$7^{n+1} \equiv 7^n \cdot 7 \equiv 7 \cdot 7 \equiv 49 \mod 8 = 1$$

We have shown that

$$7^{n+1} = \begin{cases} 1 & \text{if } 2|(n+1) \\ 7 & \text{if } n+1 \text{ is odd.} \end{cases}$$

Inductive Principle gives the desired claim ∎

Therefore

$$6^n + 7^n \equiv \begin{cases} 1+1 \mod 8, & n=0 \\ 6+7 \mod 8, & n=1 \\ 4+1 \mod 8, & n=2 \\ 1, & n>2 \text{ and } n \text{ even} \\ 7, & n>2 \text{ and } n \text{ odd} \end{cases}$$

or

$$\boxed{6^n + 7^n \equiv \begin{cases} 2, & n=0 \\ 5, & n=1 \text{ or } n=2 \\ 1, & n>2 \text{ is even} \\ 7, & n>2 \text{ is odd} \end{cases}}$$

4 **Is $\exp(\lfloor \ln(n) \rfloor) \in \Theta(n)$? Prove or disprove. Recall that $\lfloor x \rfloor$ is obtained by rounding x down to the nearest integer.**

$$\exp(\lfloor \ln n \rfloor) \in \Theta(n)$$

<u>Prf</u>

To show this, we need to find $C, c \geq 0$ so that for all $n \geq 1$,

$$cn \leq \exp(\lfloor \ln n \rfloor) \leq Cn,$$

or equivalently,

$$ce^{\ln n} \leq e^{\lfloor \ln n \rfloor} \leq Ce^{\ln n},$$

or equivalently

$$c \leq e^{\lfloor \ln n \rfloor - \ln n} \leq C.$$

To find such c, C, note that for all $n \geq 1$,

$$(\ln n - 1) - \ln n \leq \lfloor \ln n \rfloor - \ln n \leq \ln n - \ln n,$$

or equivalently

$$-1 \leq \lfloor \ln n \rfloor - \ln n \leq 0.$$

Exponentiating these inequalities, we have that

$$e^{-1} \leq e^{\lfloor \ln n \rfloor - \ln n} \leq 1$$

so we have found $C = 1$ and $c = e^{-1}$ that satisfy

$$cn \leq \exp(\lfloor \ln n \rfloor) \leq Cn$$

for all $n \geq 1$.

$\blacksquare$

5 Define the sequence of nested roots $\sqrt{m}, \sqrt{m+\sqrt{m}}, \ldots$ by the recurrence $x_{n+1} = \sqrt{m+\sqrt{x_n}}$ for $n \geq 1$, with the base case $x_1 = \sqrt{m}$. Prove that, if $n \geq 1$, then

$$x_n < \sqrt{m+\frac{1}{4}} + \frac{1}{2}.$$

Hint: prove an upper bound on x_n^2.

TYPO should be $x_{n+1} = \sqrt{m + x_n}$ to match the given sequence

Prf (with $x_{n+1} = \sqrt{m + \sqrt{x_n}}$)

The claimed bound is equivalent to

$$x_n^2 < \left(\sqrt{m+\frac{1}{4}} + \frac{1}{2}\right)^2.$$

We prove this via induction.

Base case: When $n=1$, $x_n^2 = (\sqrt{m})^2 < \left(\sqrt{m+\frac{1}{4}} + \frac{1}{2}\right)^2$.

Inductive step Assume that $x_n^2 < \left(\sqrt{m+\frac{1}{4}} + \frac{1}{2}\right)^2$.

Then

$$x_{n+1} = \sqrt{\sqrt{x_n}+m} < \sqrt{x_n+m}$$

(if $x_{n+1} = \sqrt{m+x_n}$, then directly $x_{n+1}^2 = x_n + m$)

so $$x_{n+1}^2 \leq x_n + m < \left(\sqrt{m+\frac{1}{4}} + \frac{1}{2}\right) + m$$

and note that

$$\left(\sqrt{m+\frac{1}{4}} + \frac{1}{2}\right)^2 = m + \frac{1}{4} + 2 \cdot \frac{1}{2}\sqrt{m+\frac{1}{4}} + \frac{1}{4}$$
$$= \sqrt{m+\frac{1}{4}} + \frac{1}{2} + m.$$

Thus we have that

$$x_{n+1}^2 < \left(\sqrt{m+\frac{1}{4}} + \frac{1}{2}\right)^2.$$

The claim follows from the principle of induction. $\blacksquare$

6 Let m and n be two nonnegative integers not both zero. The least common multiple $\mathbf{lcm}(m,n)$ is the smallest nonnegative integer divisible by both m and n. Prove that $\mathbf{lcm}(m,n) = \frac{mn}{\mathbf{gcd}(m,n)}$.

<u>Prf</u> We use a direct proof. For convenience let $g = \gcd(m,n)$ and write $m = gd$ and $n = gd'$. Then $\ell = \frac{mn}{g} = gdd'$ is our claimed lcm. Note that $\ell = md' = nd$ is a common multiple of m and n, so it suffices to show that any common multiple c of m and n is divisible by ℓ to establish that ℓ is indeed $\operatorname{lcm}(m,n)$.

To show this, we note first that d and d' are coprime (otherwise we could absorb their common factors into g to get a larger common divisor). This means in particular that any number divisible by d and also divisible by d' must be divisible by dd': this is established in in-text exercise 10.7(b).

Now let c be a common multiple of m and n, then for some x, $c = mx = gdx$, so $\frac{c}{g} = dx$, showing $d \mid \left(\frac{c}{g}\right)$. Similarly, $d' \mid \left(\frac{c}{g}\right)$. This implies that $dd' \mid \left(\frac{c}{g}\right)$, and therefore $gdd' \mid c$. Recall that $\ell = gdd'$, so we have shown $\ell \mid c$ for any common multiple c of m and n, which suffices to establish that $\ell = \frac{mn}{\gcd(m,n)} = \operatorname{lcm}(m,n)$.

∎

SCRATCH

SCRATCH