

MIDTERM: 120 Minutes

NO COLLABORATION or electronic devices. Any violations result in an **F**.

NO questions allowed during the test. Interpret and do the best you can.

You **MUST** show **CORRECT** work, even on multiple choice questions, to get credit.

GOOD LUCK!

1	2	3	4	5	6	Total
100	20	20	20	20	20	200

1 Circle one answer per question. 10 points for each correct answer.

- (a) The moon goes through its phases periodically in the following order: new, waxing crescent, first quarter, waxing gibbous, full, waning gibbous, third quarter, and waning crescent, then new again. Today the moon is in its waning crescent phase. What phase will the moon be in 10^{21} phases from now?

☐ A new.
☐ B waxing crescent.
☐ C first quarter.
☐ D full.
☐ E None of the above.

- (b) How many natural numbers less than 21 are coprime with 21?

☐ A 5.
☐ B 9.
☐ C 10.
☐ D 12.
☐ E None of the above.

- (c) Let A be the adjacency matrix of C_{2n} . Which of the following are true of A ?

☐ A $4 \mid \left(\sum_{i=1}^{2n} \sum_{j=1}^{2n} A_{ij} \right)$.
☐ B $\sum_{i=1}^n A_{ii} = 0$.
☐ C Each row of A sums to 2.
☐ D All three of the above options.
☐ E Two of the above options.

- (d) The negation of “The reaction to every action is equal and opposite” is:

☐ A “There are actions whose reactions are not equal and not opposite”.
☐ B “There are actions whose reactions are either not equal or not opposite”.
☐ C “For every action, the reaction is not equal and not opposite”.
☐ D “For every action, the reaction is either not equal or not opposite”.
☐ E None of the above.

- (e) If $T_0 = 1$, $T_1 = 2$, and $T_{n+2} = T_{n+1} + 5T_n$, what is the value of T_4 ?

☐ A 12
☐ B 17
☐ C 38
☐ D 52
☐ E None of the above

(f) What is $3^{2015} \bmod 7$?

- ☐ A 2
- ☐ B 3
- ☐ C 5
- ☐ D 6
- ☐ E None of the above

(g) Consider the degree sequence $[6, 6, 5, 4, 3, 3, 1]$. Which of the following is true?

- ☐ A This degree sequence is not graphical.
- ☐ B This sequence is graphical, and such a graph is disconnected.
- ☐ C This sequence is graphical, can be realized with a planar graph, and such a graph has 9 faces.
- ☐ D This sequence is graphical, and such a graph is a tree.
- ☐ E None of the above.

(h) Which claim below is true?

- ☐ A $f \in o(g) \rightarrow f \in O(g)$.
- ☐ B $f \in \Theta(g) \rightarrow g \in \Theta(f)$.
- ☐ C $f \in \omega(g) \rightarrow g \in O(f)$.
- ☐ D None of these claims are true.
- ☐ E All of these claims are true.

(i) Which of the following asymptotic relationships is correct?

- ☐ A $(n+1)^{n+1} \in O(n^n)$.
- ☐ B $(n+1)^{n+1} \in \omega(n^n)$.
- ☐ C $(n+1)^{n+1} \in o(n^n)$.
- ☐ D $(n+1)^{n+1} \in \Theta(n^n)$.
- ☐ E None of the above.

(j) How many non-isomorphic connected acyclic graphs exist that have four vertices?

- ☐ A 1.
- ☐ B 2.
- ☐ C 3.
- ☐ D 4.
- ☐ E More than 4.

2 Recall the equation for integration by parts, $\int f dg = fg - \int g df$. The formula for summation by parts is

$$\sum_{i=m}^n f_i(g_{i+1} - g_i) = (f_n g_{n+1} - f_m g_m) - \sum_{i=m+1}^n g_i(f_i - f_{i-1}).$$

Choose appropriate sequences f_i and g_i and use summation by parts to show that

$$\sum_{i=0}^n i2^i = (n-1)2^{n+1} + 2.$$

3 What is the remainder when $6^n + 7^n$ is divided by 8?

- 4** Is $\exp(\lfloor \ln(n) \rfloor) \in \Theta(n)$? **Prove or disprove.** Recall that $\lfloor x \rfloor$ is obtained by rounding x down to the nearest integer.

- 5** Define the sequence of nested roots $\sqrt{m}, \sqrt{m + \sqrt{m}}, \dots$ by the recurrence $x_{n+1} = \sqrt{m + x_n}$ for $n \geq 1$, with the base case $x_1 = \sqrt{m}$. Prove that, if $n \geq 1$, then

$$x_n < \sqrt{m + \frac{1}{4}} + \frac{1}{2}.$$

Hint: prove an upper bound on x_n^2 .

- 6** Let m and n be two nonnegative integers not both zero. The least common multiple $\text{lcm}(m, n)$ is the smallest nonnegative integer divisible by both m and n . Prove that $\text{lcm}(m, n) = \frac{mn}{\text{gcd}(m, n)}$.

SCRATCH

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