FINAL v1: 180 Minutes

Answer **ALL** questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

You **MUST** show **CORRECT** work to get full credit.

When in doubt, TINKER.

1	2	3	4	5	6	Total
200	30	30	30	30	30	350

1 Circle one answer per question. 10 points for each correct answer.

(1) Consider greedy coloring on a graph with n vertices. Which of the following is true about the number of colors c that will be used to color the graph?

- [A] $c \le \chi(G)$
- [B] $c \ge \max_{i=1}^{n} \delta_i + 1$
- [C] $c \le \chi(G) + 1$
- [~~D~~] $c \le \max_{i=1}^{n} \delta_i + 1$
- [E] Not enough information

(this obs was made in-lecture) we know when node i is reached, at most δ_i colors have been used to color its neighbors, so color δ_i+1 is the largest color used to color it

(2) How many ways can 7 men and 7 women in a dance class be paired into couples of different genders?

- [~~A~~] 7!
- [B] $(7!)^2$
- [C] $\binom{7}{2}$
- [D] $\binom{7}{2}^2$
- [E] None of the above

Order the 7 men $1, \ldots, 7$. Then you have $7!$ ways of pairing the women with men

(3) How many 7-bit sequences begin with 1 or contain 100 starting at position 3?

- [A] 16
- [B] 64
- [~~C~~] 72
- [D] 80
- [E] None of the above

inclusion-exclusion: $2^6 + 2^4 - 2^3 = 72$

2^6: # 7 bit beginning w/ 1

2^4: # 7 bit containing 100 starting at pos 3

2^3: # 7 bit beg w/ 1 and contain 100 at pos 3

(4) 13 cards are dealt randomly from a standard 52-card deck. Compute the probability of getting three Aces.

- [~~A~~] $\binom{4}{3}\binom{48}{10}/\binom{52}{13}$
- [B] $\binom{52}{4}/\binom{52}{13}$
- [C] $30/\binom{52}{13}$
- [D] $(4/52)^3\binom{13}{3}/\binom{52}{13}$
- [E] None of the above

$\mathbb{P}$(choose any 3 of the 4 aces, then 10 of the remaining non-ace cards)

$$= \frac{\binom{4}{3}\binom{52-4}{10}}{\binom{52}{13}} = \frac{\binom{4}{3}\binom{48}{10}}{\binom{52}{13}}$$

(5) Jianyou and Rishi are randomly positioned in a line of five total people waiting to buy movie tickets. What is the probability that Jianyou and Rishi are adjacent to each other in line?

- [A] 1/5
- [~~B~~] 2/5
- [C] 1/25
- [D] 1/20
- [E] None of the above

$$\mathbb{P}(J \,\&\, R \text{ are adjacent}) = \sum_{i=1}^{4} \mathbb{P}(J \,\&\, R \text{ adj and start at pos } i)$$

$$= \sum_{i=1}^{4} \left[\mathbb{P}(J=i \,\&\, R=i+1) + \mathbb{P}(J=i+1 \,\&\, R=i)\right]$$

$$= 4 \cdot \left(\frac{1}{5}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{1}{4}\right) = 2/5$$

(6) What is the contrapositive of "If xy is even, then x is even or y is even"?

$p \to q$ has contrapositive $\neg q \to \neg p$

- [A] If xy is odd, then x is odd or y is odd.
- [B] If xy is odd, then x is odd and y is odd.
- [C] If x is odd and y is odd, then xy is odd.
- [D] If x is odd or y is odd, then xy is odd.
- [E] None of the above

(7) The independent random variables **X** and **Y** have the same pdf, $p(k) = 2^{-k}$ for $k \in \mathbb{N}$. Compute $\mathbb{P}[\mathbf{X} = \mathbf{Y}]$.

- [A] 1/6
- [B] 1/3
- [C] 1/2
- [D] 2/3
- [E] 4/5

$$\mathbb{P}(X=Y) = \mathbb{P}[X=Y=1] + \cdots + \mathbb{P}[X=Y=10^{100}] + \cdots$$

$$= \sum_{k=1}^{\infty} \mathbb{P}[X=k]\,\mathbb{P}[Y=k] = \sum_{k=1}^{\infty} 2^{-2k} = \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k$$

$$= \frac{1}{1-\frac{1}{4}} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$$

(8) Let $f(n) = \ln(n^2) + \ln^2(n)$. Which of the following asymptotic relations is most precise?

- [A] $f(n) \in \Omega(\ln(n))$
- [B] $f(n) \in \omega(\ln(n))$
- [C] $f(n) \in o(\ln(n))$
- [D] $f(n) \in \Theta(\ln(n))$
- [E] $f(n) \in O(\ln(n))$

$$\frac{f(n)}{\ln(n)} > \frac{\ln^2(n)}{\ln(n)} = \ln(n) \to \infty, \text{ so}$$

$$f(n) \in \omega(\ln(n))$$

(9) In which of the domains $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ is it true that $\forall y : (\exists x : x^2 = y)$?

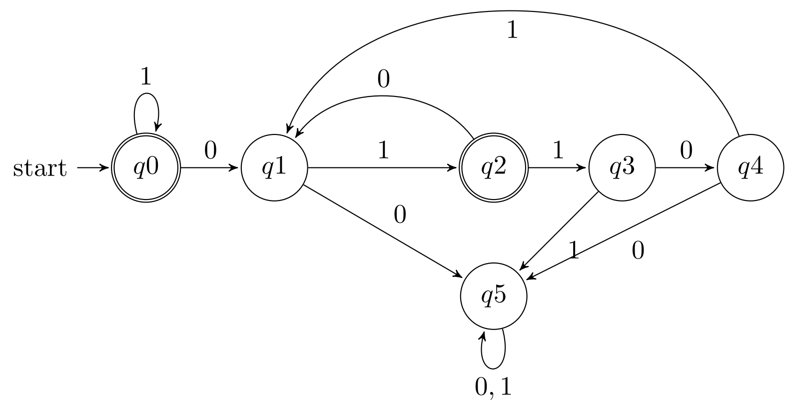
everyone gets credit

- [A] $\mathbb{Q}$
- [B] $\mathbb{R}$
- [C] $\mathbb{Q}$ and $\mathbb{R}$
- [D] $\mathbb{N}$ and $\mathbb{Z}$
- [E] $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$

actually none due to (square roots always exist only in $\mathbb{R}_+$)

(10) Which string *cannot* be generated by the CFG $S \to \varepsilon \,|\, 0 \,|\, 1 \,|\, S0S$?

- [A] 1010 ✓ $\varepsilon \to S0S \to 10S \to 10S0S \to 1010S \to 1010\varepsilon$
- [B] 1001 ✓ $\varepsilon \to S0S \to 10S \to 10S0S \to 10S01 \to 10\varepsilon 01$
- [C] 01101 no way to generate adjacent ones with this grammar
- [D] 0010 ✓ $\varepsilon \to S0S \to 00S \to 00S0S \to 0010S \to 0010\varepsilon$
- [E] All of the above can be generated



(11) Which string is accepted by this DFA?

- A 1011
- B 101000
- C 101101
- D 110011
- ~~E~~ 1110101

ends at q2

(12) What is the coefficient of x^7 in $(\sqrt{x} + 5x)^{10}$?

- A $\binom{10}{4}5^6$
- B $\binom{10}{5}5^4$
- ~~C~~ $\binom{10}{6}5^4$
- D $\binom{10}{7}5^3$
- E $\binom{10}{7}5^7$

$$= \sum_{k=0}^{10} \binom{10}{k} (5x)^k (\sqrt{x})^{10-k}$$

$$= \sum_{k=0}^{10} \binom{10}{k} 5^k x^k x^{5-k/2} = \sum_{k=0}^{10} \binom{10}{k} 5^k x^{5+k/2}$$

$5 + k/2 = 7 \Rightarrow k = 4$

so coeff of x^7 is $\binom{10}{4}5^4$

$= \binom{10}{6}5^4$

(13) Which is logically equivalent to $(p \wedge \neg q) \rightarrow r$?

- A $(\neg p \wedge q) \vee r$
- B $(\neg p \vee \neg q) \vee r$
- C $\neg r \rightarrow p \vee \neg q$
- ~~D~~ $\neg r \rightarrow \neg p \vee q$
- E $\neg r \rightarrow \neg p \wedge q$

contrapositive

(14) Two garages have 200 cars each; cars are either red or white. The first garage has exactly 25 red cars and the second garage has exactly 10 red cars. You were asleep when your friend pulled into a randomly chosen garage. You look around and the first car you randomly see is red. What is the probability that you are in the first garage?

$$\mathbb{P}(\text{garage 1} \mid \text{red}) = \frac{\mathbb{P}(\text{garage 1 \& red})}{\mathbb{P}(\text{red})}$$

- A 1/2
- B 3/5

$$= \frac{\mathbb{P}(\text{red} \mid \text{garage 1})\,\mathbb{P}(\text{garage 1})}{\mathbb{P}(r \mid 1)\,\mathbb{P}(1) + \mathbb{P}(r \mid 2)\,\mathbb{P}(2)}$$

$\boxed{\text{C}}$ 4/5

$\cancel{\boxed{\text{D}}}$ 5/7

$\boxed{\text{E}}$ None of the above

$$= \frac{\frac{25}{200}\cdot\frac{1}{2}}{\frac{25}{200}\cdot\frac{1}{2}+\frac{10}{200}\cdot\frac{1}{2}} = \frac{25}{25+10} = \frac{25}{35} = \frac{5}{7}$$

(15) Reggie has fifty textbooks, and a bookshelf with six shelves. How many ways can Reggie arrange his books on these shelves if the order of the books on each shelf does not matter, and each shelf can hold all fifty books?

$\boxed{\text{A}}$ 50^6

$\cancel{\boxed{\text{B}}}$ 6^{50}

$\boxed{\text{C}}$ $\binom{50}{6}$

$\boxed{\text{D}}$ $50!/6!$

$\boxed{\text{E}}$ None of the above

for each of the 50 books we have 6 choices of shelf to place it on

(16) Evaluate the sum $T(n) = \sum_{i=1}^{n}(3i + 3^{-i})$. $= \frac{3n(n+1)}{2} + \sum_{i=0}^{n}\left(\frac{1}{3}\right)^i - 1$

$\cancel{\boxed{\text{A}}}$ $\frac{1}{2}\left(3n^2 + 3n - 3^{-n} + 1\right)$

$\boxed{\text{B}}$ $\frac{1}{2}\left(3n^2 + 3n - 3^{-n} + 3\right)$

$\boxed{\text{C}}$ $\frac{1}{2}\left(3n^2 - 2n - 3^{-n} + 1\right)$

$\boxed{\text{D}}$ $\frac{1}{2}\left(3n^2 - 2n - 3^{-n} + 3\right)$

$\boxed{\text{E}}$ None of the above

$$= \frac{3n(n+1)}{2} + \frac{1-(1/3)^{n+1}}{1-1/3} - 1 = \frac{3n(n+1)}{2} + \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^{n+1}\right) - 1$$

$$= \frac{1}{2}\left(3n^2 + 3n + 3 - \left(\frac{1}{3}\right)^n - 2\right) = \frac{1}{2}\left(3n^2+3n+1-\left(\frac{1}{3}\right)^n\right)$$

(17) If $\mathcal{L} \leq_R \mathcal{L}_{\text{EXP}}$, which of the following is true?

(I) $\mathcal{L}$ is decidable ← true b/c $\mathcal{L}_{EXP}$ is decidable

(II) $\mathcal{L}$ is in the class $\mathcal{P}$ (the set of efficiently solvable problems) ← false b/c $\mathcal{L}_{EXP} \notin P$ by design

(III) $\mathcal{L}$ is recognizable ← true b/c decidable implies recognizable

$\boxed{\text{A}}$ I only

$\boxed{\text{B}}$ II only

$\boxed{\text{C}}$ III only

$\cancel{\boxed{\text{D}}}$ I and III only

$\boxed{\text{E}}$ I, II, and III

(18) If $P(2)$ is true and $P(n) \to (P(n^2) \wedge P(n-2))$ for $n \geq 2$, which of the following can I conclude?

$\cancel{\boxed{\text{A}}}$ $P(120)$ is true

$\boxed{\text{B}}$ $P(46)$ is not true

$\boxed{\text{C}}$ $P(1)$ is not true

$\boxed{\text{D}}$ $P(11)$ is true

$\boxed{\text{E}}$ None of the above

from implications we see $P(n)$ true when n even

(says nothing about whether $P(n)$ is true or false when n is odd)

(19) Which of these degree sequences is graphical?

A $[3, 2, 2, 2]$

B $[3, 3, 3, 1]$

C $[3, 3, 3, 2]$

~~D~~ $[3, 3, 3, 3]$ ← K_4 has this degree sequence

E None of the above

(20) Which set is *not* countable?

~~A~~ The set of subsets of $\mathbb{N}$. ← can get a bijection to set of infinite binary sequences, so uncountable

B The set of all possible FOCS finals. ← each is a finite length bit string in ASCII, and Σ^* is countable

C The set of eventually constant infinite sequences on $\mathbb{Z}$. ← from a HW question, this is countable

D The Cartesian product of two countable sets. ← we know $A \times B$ is countable if A, B are by the tabular argument used to show $\mathbb{N} \times \mathbb{Z}$ is countable

E The set of finite subsets of $\mathbb{Q}$. ← can get an injection to Σ^*, so countable

2 **There are initially n pairs of chopsticks, then m of the chopsticks randomly break during transportation. We say that a pair arrives intact if neither of its chopsticks are broken. What is the expected number of pairs that arrive intact?** ***Hint: take $\mathbf{X}_i = 1$ if the ith pair arrives intact.***

Let X be the number of pairs that arrive intact,

$$X = \sum_{i=1}^{n} X_i .$$

Then

$$\mathbb{E}X = \sum_{i=1}^{n} \mathbb{E}X_i .$$

Note that $X_i \sim \text{Bern}(p)$, where

$$p_i = \mathbb{P}(X_i = 1)$$

$$= \mathbb{P}(\text{none of the chopsticks in the } i\text{th pair break})$$

$$= \frac{\binom{2n-2}{m}}{\binom{2n}{m}} = \frac{(2n-2)!}{m!\,(2n-m-2)!} \quad \frac{m!\,(2n-m)!}{2n!}$$

$$= \frac{1}{2n(2n-1)} \quad (2n-m)(2n-m-1) \equiv p$$

Since $\mathbb{E}X_i = p_i = p$, we have

$$\boxed{\mathbb{E}X = np = \frac{1}{2} \frac{(2n-m)(2n-m-1)}{(2n-1)}}$$

3 If $\mathcal{L}$ and $\overline{\mathcal{L}}$ are *recognizable*, show that $\mathcal{L}$ is decidable by sketching a *decider* for $\mathcal{L}$.

Let R_1 and R_2 be recognizers for $\mathcal{L}$ and $\overline{\mathcal{L}}$ respectively. Here is the pseudo-code for a decider M for $\mathcal{L}$:

Input: $\omega \in \Sigma^*$

Procedure: Use a universal Turing machine to run $R_1(\omega)$ and $R_2(\omega)$ in parallel.

- If R_1 finishes first and accepts, or R_2 finishes first and rejects, then ACCEPT
- If R_1 finishes first and rejects, or R_2 finishes first and accepts, then REJECT.

To prove that this is a decider for $\mathcal{L}$, consider the cases:

1) $\omega \in \mathcal{L}$: then R_1 will halt and ACCEPT because it is a recognizer for $\mathcal{L}$. If R_2 halts and REJECTs first, then M returns ACCEPT as it should. Otherwise R_1 will halt first and ACCEPT first, so M will ACCEPT as it should. In either case, M accepts ω.

2) $w \notin \mathcal{L}$: then $\underline{R_2}$ will halt and ACCEPT because $w \in \overline{\mathcal{L}}$. If R_2 halts and ACCEPTs first, then M will REJECT as it should. Otherwise R_1 will halt and REJECT first, so M will REJECT as it should. In either case, M rejects w.

Since M halts and ACCEPTs $w \in \mathcal{L}$ and halts and rejects $w \notin \mathcal{L}$, M is a decider for $\mathcal{L}$.

4 **Consider the repetition language $\mathcal{L} = \{\omega\omega \,|\, \omega \in \{0,1\}^*\}$. Show that this problem is in $\mathcal{P}$ (the class of efficiently solvable problems) by giving pseudocode for a decider and analyzing the worst-case runtime.**

Take the size of the input to be the length of half of $\omega\omega$, called n. Recall that a problem is in $\mathcal{P}$ if there is a decider for that problem that runs in time polynomial in n on any reasonable Turing architecture. For simplicity we use a Turing machine with two tapes.

PSEUDOCODE for M

Setup

1. Check that the input is well-formed (has an even-number of bits) and return to start of input on tape 1.
2. Copy input to tape 2 and return to start of both tapes.
3. If the first character is ␣, accept

Find the middle of the input and mark it on tape 2

4. Move the tape heads simultaneously to the right until you encounter the first unmarked bit on tape 1. If there is no unmarked bit on tape 1, mark the bit to the right of the current position on tape 2, and GOTO step 5.
   Otherwise mark this bit on tape 1 and move to the right until you encounter the last unmarked bit on tape 1. If you encounter ␣ before a marked bit, move the heads back by one bit and mark this bit on tape 1.
   Move both heads back to the start of input and GOTO step 4.
5. Move to the start of input on both tapes, and overwrite tape 1 with the contents of tape 2, not including the mark. Return to the start of both tapes

Check the two halves of the input match

6. Move to the mark on tape 2, and the first input character on tape 1. If these bits do not match, REJECT.

7. Move once to the right on both tapes. If the character on tape 2 is ␣, ACCEPT. If the characters on the two tapes do not match, REJECT. Otherwise GOTO step 7

Now we analyze the worst-case runtime. One worst case input of size n is the string consisting of 2n ones, because both halves of the string must be checked completely before accepting. With this input:

- Setup takes $\Theta(n)$ operations
- Step 4 takes $\Theta(n)$ to mark bits at both ends of the string and return to the start. This must be done $\Theta(n)$ times. Step 5 takes $\Theta(n)$ ops. Overall, Finding and marking the middle takes $\Theta(n^2)$ operations
- Checking the two halves match takes $\Theta(n)$ operations

Thus $T(n) = \Theta(n^2)$ for this decider, which shows this problem is in P.

5 **Consider the set $\mathcal{F}$ recursively defined as follows: (i) $1 \in \mathcal{F}$, (ii) $f \in \mathcal{F} \to \omega + 1/f \in \mathcal{F}$ for** $\omega \in \{0, 1, 2, \ldots\}$. ← we call this $\mathbb{N}_0$

Prove that $\mathcal{F}$ is the set of positive rational numbers: ← we call this $\mathbb{Q}_+$

(i) Prove that every element in $\mathcal{F}$ is a positive rational number.

(ii) Prove that every positive rational number is in $\mathcal{F}$: specifically, let $x = \frac{a}{b}$ where $a, b \in \mathbb{N}$, and show that $x \in \mathcal{F}$ by strong induction on b. For the base case take $b = 1$ and prove that $a \in \mathcal{F}$ for all $a \in \mathbb{N}$. For the induction step, assume $\frac{a}{b} \in \mathcal{F}$ for $b \in \{1, \ldots, n\}$ and $a \in \mathbb{N}$, then show that $\frac{a}{n+1} \in \mathcal{F}$.

Hint: the quotient remainder theorem is helpful in the induction step.

Prf

(i) We use structural induction to establish that if $x \in \mathcal{F}$, then $x \in \mathbb{Q}_+$. The base case $1 \in \mathbb{Q}_+$. Now assume that $f \in \mathcal{F}$ also satisfies $f \in \mathbb{Q}_+$, then $\frac{1}{f} \in \mathbb{Q}_+$, so for any $\omega \in \{0, 1, 2, \ldots\}$, we have that $\omega + \frac{1}{f} \in \mathbb{Q}_+$. By structural induction, $\mathcal{F} \subseteq \mathbb{Q}_+$.

(ii) We use strong induction to establish that $\mathbb{Q}_+ \subseteq \mathcal{F}$. For the base cases, note that $1 \in \mathcal{F}$, so using the constructor rule we have that $1 + \omega \in \mathcal{F}$ for any $\omega \in \mathbb{N}_0$. Since $\mathbb{N} = \{x = 1 + \omega \mid \omega \in \mathbb{N}_0\}$, we have that $\mathbb{N} \subseteq \mathcal{F}$. Thus for any $x \in \mathbb{Q}_+$ of the form $x = \frac{a}{1}$, we have that $x \in \mathcal{F}$.

For the inductive step, assume that $x = \frac{a}{b} \in \mathcal{F}$ when $b \in \{1, \ldots, n\}$ and $a \in \mathbb{N}$. Fix an $a \in \mathbb{N}$. By the quotient remainder theorem, $a = q \cdot (n+1) + r$ where $q \in \mathbb{N}$ and $r \in \mathbb{N}$ satisfies $0 \leq r \leq n$. Therefore,

$$\frac{a}{n+1} = q + \frac{r}{n+1} = q + \left(\frac{n+1}{r}\right)^{-1}.$$

We showed that $\mathbb{N} \subseteq \mathcal{F}$, so $q \in \mathcal{F}$.

We also have that $\frac{n+1}{r}$ takes the form $\frac{a}{b}$ where $a \in \mathbb{N}$ and $b \in \{1, \ldots, n\}$, so by the inductive hypothesis $\frac{n+1}{r} \in \mathcal{F}$. By the constructor rule for $\mathcal{F}$,

$$\frac{a}{n+1} = q + \left(\frac{n+1}{r}\right)^{-1} \in \mathcal{F}.$$

By the principle of strong induction, we conclude that all numbers of the form $\frac{a}{b}$ where $a \in \mathbb{N}$ and $b \in \mathbb{N}$ are in $\mathcal{F}$.

This is exactly $\mathbb{Q}_+$, so we have established that

$$\mathbb{Q}_+ = \mathcal{F}.$$

6 **Suppose x^2 is a multiple of y for integers $x, y > 1$. Show that $\gcd(x, y) > 1$.** ***Hint: use prime factorization or Bezout's Identity.***

Prf (Bezout's Identity)

We use proof by contradiction. Assume $y|x^2$ and $\gcd(x,y) = 1$, then by Bezout's Identity there are integers m and n satisfying

$$mx + ny = 1.$$

It follows that $mx^2 + nyx = x$. By assumption $y|x^2$, so $y|(mx^2 + nyx)$. Thus $y|x$ and we have that $\gcd(x,y) = y > 1$, a contradiction. Therefore either $y \nmid x^2$ or $\gcd(x,y) > 1$. ∎

Prf (Prime factorization)

Assume that $y|x^2$. We use a direct proof. By the prime fact. theorem (fundamental theorem of arithmetic), there is a a unique factorization of x as a product of powers of primes. That is,

$$x = a_1^{p_1} a_2^{p_2} \cdots a_k^{p_k}$$

for some $k \geq 1$, prime numbers $a_1 \ldots a_k$, and $p_1 \cdots p_k$ in $\mathbb{N}$. Thus,

$$x^2 = a_1^{2p_1} a_2^{2p_2} \cdots a_k^{2p_k}.$$

Since y divides this number, it does not contain any prime other than $a_1, \ldots, a_k$ (otherwise by writing x as a product involving the prime factorization of y, we would see that x contains other primes than $a_1, \ldots, a_k$ which contradicts the fundamental theorem of arithmetic).

That is, $y = a_1^{q_1} a_2^{q_2} \cdots a_k^{q_k}$

where $q_1, \dots, q_k \in \mathbb{N}$ satisfy $0 \le q_i \le 2p_i$.

By assumption $y > 1$, so there is some i so that q_i is nonzero.

It follows that $\gcd(x, y) \geq a_i^{q_i} > 1$.

$\blacksquare$

SCRATCH

SCRATCH