FINAL v1: <u>180 Minutes</u>

Answer **ALL** questions.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

You **MUST** show **CORRECT** work to get full credit.

When in doubt, TINKER.

1	2	3	4	5	6	Total
200	30	30	30	30	30	350

1 Circle one answer per question. 10 points for each correct answer.

(1) Consider greedy coloring on a graph with n vertices. Which of the following is true about the number of colors c that will be used to color the graph?

$\boxed{\text{A}}$ $c \leq \chi(G)$

$\boxed{\text{B}}$ $c \geq \max_{i=1}^{n} \delta_i + 1$

$\boxed{\text{C}}$ $c \leq \chi(G) + 1$

$\boxed{\text{D}}$ $c \leq \max_{i=1}^{n} \delta_i + 1$

$\boxed{\text{E}}$ Not enough information

(2) How many ways can 7 men and 7 women in a dance class be paired into couples of different genders?

$\boxed{\text{A}}$ $7!$

$\boxed{\text{B}}$ $(7!)^2$

$\boxed{\text{C}}$ $\binom{7}{2}$

$\boxed{\text{D}}$ $\binom{7}{2}^2$

$\boxed{\text{E}}$ None of the above

(3) How many 7-bit sequences begin with 1 or contain 100 starting at position 3?

$\boxed{\text{A}}$ 16

$\boxed{\text{B}}$ 64

$\boxed{\text{C}}$ 72

$\boxed{\text{D}}$ 80

$\boxed{\text{E}}$ None of the above

(4) 13 cards are dealt randomly from a standard 52-card deck. Compute the probability of getting three Aces.

$\boxed{\text{A}}$ $\binom{4}{3}\binom{48}{10}/\binom{52}{13}$

$\boxed{\text{B}}$ $\binom{52}{4}/\binom{52}{13}$

$\boxed{\text{C}}$ $30/\binom{52}{13}$

$\boxed{\text{D}}$ $(4/52)^3\binom{13}{3}/\binom{52}{13}$

$\boxed{\text{E}}$ None of the above

(5) Jianyou and Rishi are randomly positioned in a line of five total people waiting to buy movie tickets. What is the probability that Jianyou and Rishi are adjacent to each other in line?

$\boxed{\text{A}}$ 1/5

$\boxed{\text{B}}$ 2/5

$\boxed{\text{C}}$ 1/25

$\boxed{\text{D}}$ 1/20

$\boxed{\text{E}}$ None of the above

(6) What is the contrapositive of "If xy is even, then x is even or y is even"?

$\boxed{\text{A}}$ If xy is odd, then x is odd or y is odd.

$\boxed{\text{B}}$ If xy is odd, then x is odd and y is odd.

$\boxed{\text{C}}$ If x is odd and y is odd, then xy is odd.

$\boxed{\text{D}}$ If x is odd or y is odd, then xy is odd.

$\boxed{\text{E}}$ None of the above

(7) The independent random variables $\mathbf{X}$ and $\mathbf{Y}$ have the same pdf, $p(k) = 2^{-k}$ for $k \in \mathbb{N}$. Compute $\mathbb{P}[\mathbf{X} = \mathbf{Y}]$.

$\boxed{\text{A}}$ $1/6$

$\boxed{\text{B}}$ $1/3$

$\boxed{\text{C}}$ $1/2$

$\boxed{\text{D}}$ $2/3$

$\boxed{\text{E}}$ $4/5$

(8) Let $f(n) = \ln(n^2) + \ln^2(n)$. Which of the following asymptotic relations is most precise?

$\boxed{\text{A}}$ $f(n) \in \Omega(\ln(n))$

$\boxed{\text{B}}$ $f(n) \in \omega(\ln(n))$

$\boxed{\text{C}}$ $f(n) \in o(\ln(n))$

$\boxed{\text{D}}$ $f(n) \in \Theta(\ln(n))$

$\boxed{\text{E}}$ $f(n) \in O(\ln(n))$

(9) In which of the domains $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ is it true that $\forall y : (\exists x : x^2 = y)$?

$\boxed{\text{A}}$ $\mathbb{Q}$

$\boxed{\text{B}}$ $\mathbb{R}$

$\boxed{\text{C}}$ $\mathbb{Q}$ and $\mathbb{R}$

$\boxed{\text{D}}$ $\mathbb{N}$ and $\mathbb{Z}$

$\boxed{\text{E}}$ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$

(10) Which string *cannot* be generated by the CFG $S \to \varepsilon \,|\, 0 \,|\, 1 \,|\, S0S$?

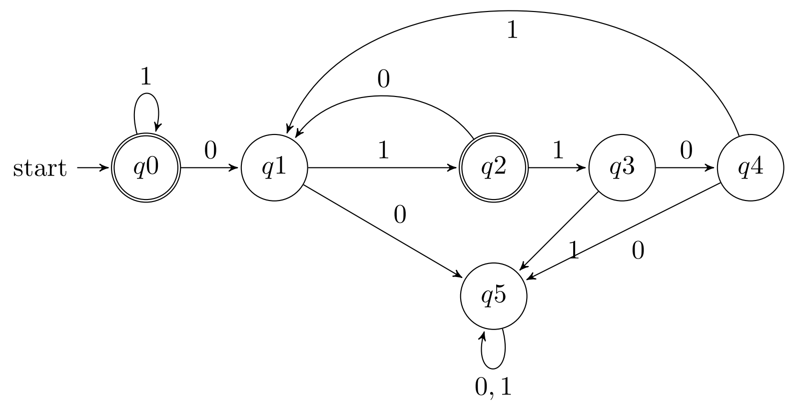
$\boxed{\text{A}}$ 1010

$\boxed{\text{B}}$ 1001

$\boxed{\text{C}}$ 01101

$\boxed{\text{D}}$ 0010

$\boxed{\text{E}}$ All of the above can be generated



(11) Which string is accepted by this DFA?

- A 1011
- B 101000
- C 101101
- D 110011
- E 1110101

(12) What is the coefficient of x^7 in $(\sqrt{x} + 5x)^{10}$?

- A $\binom{10}{4}5^6$
- B $\binom{10}{5}5^4$
- C $\binom{10}{6}5^4$
- D $\binom{10}{7}5^3$
- E $\binom{10}{7}5^7$

(13) Which is logically equivalent to $(p \wedge \neg q) \to r$?

- A $(\neg p \wedge q) \vee r$
- B $(\neg p \vee \neg q) \vee r$
- C $\neg r \to p \vee \neg q$
- D $\neg r \to \neg p \vee q$
- E $\neg r \to \neg p \wedge q$

(14) Two garages have 200 cars each; cars are either red or white. The first garage has exactly 25 red cars and the second garage has exactly 10 red cars. You were asleep when your friend pulled into a randomly chosen garage. You look around and the first car you randomly see is red. What is the probability that you are in the first garage?

- A 1/2
- B 3/5

C $4/5$

D $5/7$

E None of the above

(15) Reggie has fifty textbooks, and a bookshelf with six shelves. How many ways can Reggie arrange his books on these shelves if the order of the books on each shelf does not matter, and each shelf can hold all fifty books?

A 50^6

B 6^{50}

C $\binom{50}{6}$

D $50!/6!$

E None of the above

(16) Evaluate the sum $T(n) = \sum_{i=1}^{n}(3i + 3^{-i})$.

A $\frac{1}{2}\left(3n^2 + 3n - 3^{-n} + 1\right)$

B $\frac{1}{2}\left(3n^2 + 3n - 3^{-n} + 3\right)$

C $\frac{1}{2}\left(3n^2 - 2n - 3^{-n} + 1\right)$

D $\frac{1}{2}\left(3n^2 - 2n - 3^{-n} + 3\right)$

E None of the above

(17) If $\mathcal{L} \leq_R \mathcal{L}_{\text{EXP}}$, which of the following is true?

(I) $\mathcal{L}$ is decidable
(II) $\mathcal{L}$ is in the class $\mathcal{P}$ (the set of efficiently solvable problems)
(III) $\mathcal{L}$ is recognizable

A I only

B II only

C III only

D I and III only

E I, II, and III

(18) If $P(2)$ is true and $P(n) \rightarrow (P(n^2) \wedge P(n-2))$ for $n \geq 2$, which of the following can I conclude?

A $P(120)$ is true

B $P(46)$ is not true

C $P(1)$ is not true

D $P(11)$ is true

E None of the above

(19) Which of these degree sequences is graphical?

$\boxed{\text{A}}$ $[3, 2, 2, 2]$

$\boxed{\text{B}}$ $[3, 3, 3, 1]$

$\boxed{\text{C}}$ $[3, 3, 3, 2]$

$\boxed{\text{D}}$ $[3, 3, 3, 3]$

$\boxed{\text{E}}$ None of the above

(20) Which set is *not* countable?

$\boxed{\text{A}}$ The set of subsets of $\mathbb{N}$.

$\boxed{\text{B}}$ The set of all possible FOCS finals.

$\boxed{\text{C}}$ The set of eventually constant infinite sequences on $\mathbb{Z}$.

$\boxed{\text{D}}$ The Cartesian product of two countable sets.

$\boxed{\text{E}}$ The set of finite subsets of $\mathbb{Q}$.

2 **There are initially n pairs of chopsticks, then m of the chopsticks randomly break during transportation. We say that a pair arrives intact if neither of its chopsticks are broken. What is the expected number of pairs that arrive intact?** ***Hint: take $\mathbf{X}_i = 1$ if the ith pair arrives intact.***

3 **If $\mathcal{L}$ and $\overline{\mathcal{L}}$ are *recognizable*, show that $\mathcal{L}$ is decidable by sketching a *decider* for $\mathcal{L}$.**

4 **Consider the repetition language $\mathcal{L} = \{\omega\omega \,|\, \omega \in \{0,1\}^*\}$. Show that this problem is in $\mathcal{P}$ (the class of efficiently solvable problems) by giving pseudocode for a decider and analyzing the worst-case runtime.**

5 Consider the set $\mathcal{F}$ recursively defined as follows: (i) $1 \in \mathcal{F}$, (ii) $f \in \mathcal{F} \to \omega + 1/f \in \mathcal{F}$ for $\omega \in \{0, 1, 2, \ldots\}$.

Prove that $\mathcal{F}$ is the set of positive rational numbers:

(i) Prove that every element in $\mathcal{F}$ is a positive rational number.

(ii) Prove that every positive rational number is in $\mathcal{F}$: specifically, let $x = \frac{a}{b}$ where $a, b \in \mathbb{N}$, and show that $x \in \mathcal{F}$ by strong induction on b. For the base case take $b = 1$ and prove that $a \in \mathcal{F}$ for all $a \in \mathbb{N}$. For the induction step, assume $\frac{a}{b} \in \mathcal{F}$ for $b \in \{1, \ldots, n\}$ and $a \in \mathbb{N}$, then show that $\frac{a}{n+1} \in \mathcal{F}$.

Hint: the quotient remainder theorem is helpful in the induction step.

6 Suppose x^2 is a multiple of y for integers $x, y > 1$. Show that $\gcd(x, y) > 1$. *Hint: use prime factorization or Bezout's Identity.*

SCRATCH

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