Foundations of Computer Science
Lecture 23

Languages: What is Computation?

A Formal Model of a Computing Problem
Decision Problems and Languages
Describing a Language: Regular Expressions
Complexity of a Computing Problem

"Say what's on your mind, Harris—the language of dance has always eluded me."

Last Time

1. Comparing infinite sets.
2. Countable.
   - $\mathbb{N}_0, E, \mathbb{Z}, \mathbb{Q}$ are countable.
   - Finite binary strings $\mathcal{B}$ is countable.
3. Uncountable
   - *Infinite* binary strings are uncountable.
   - Reals are uncountable.
4. Infinity and computing.
   - Programs are finite binary strings (countable).
   - Functions we might like to compute are infinite binary strings (uncountable).
   - Conclusion: there are **MANY** functions which *cannot* be computed by programs.

Today: Languages: What is Computation?

1. Decision problems.
2. Languages.
    - Describing a language.
3. Complexity of a computing problem.

What is a Computing Problem?

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Given $n \in \mathbb{N}$, walk through this list.

1: If you come to n output $\boxed{\text{YES}}$.
2: If you come to a number bigger than n, output $\boxed{\text{NO}}$.

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$\boxed{\text{LANGUAGES}}$

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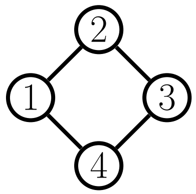
Decision problems can be formulated as testing membership in a set of strings

A Decision Problem on Graphs

(a)[**Optimization**] What's distance between nodes ① and ③? Answer: **2**



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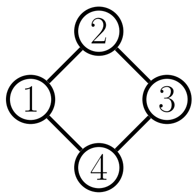

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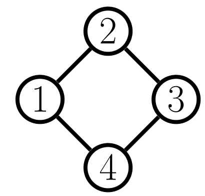

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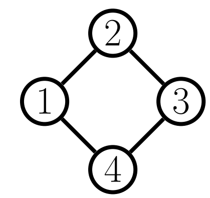

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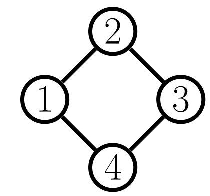

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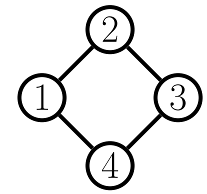

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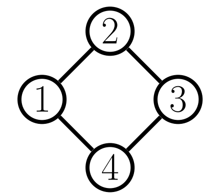

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" $1, 2, 3, 4 \mid (1, 2)(2, 3)(3, 4)(4, 1)$ "

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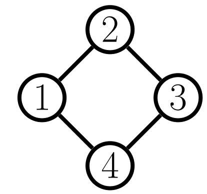

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nodes edges endpoints of path

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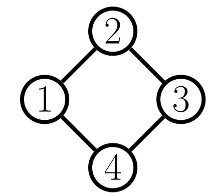

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The graph problem can be encoded as a binary string using ASCII

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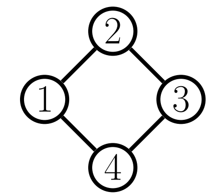

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$$\mathcal{L}_{\text{path}} = \left\{\begin{array}{l}\text{All strings of the form “nodes} \mid \text{edges} \mid \text{endpoints of path} \mid \text{target distance” for which} \\ \text{the distance between the endpoints in the graph is at most the target distance.}\end{array}\right\}$$

Pop Quiz. YES or NO: " $1,2,3,4,5 \mid (1,2)(2,3)(3,5)(3,4) \mid 1,5 \mid 2$ "

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If you can solve the decision problem, you can solve the optimization problem.

Is there a path in the graph between nodes x and y of length at most 1? $\boxed{\text{NO}}$

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Is there a path in the graph between nodes x and y of length at most **3**? NO
Is there a path in the graph between nodes x and y of length at most **4**? YES

You ask the decision question until the answer is YES.

The minimum-pathlength between x and y is 4.

It can take long, but it works.

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Decision and optimization are "equivalent" when it comes to *solvability*.

A computing problem is a decision problem.

Languages

Standard formulation of a decision problem:

Languages

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Every decision problem has a $\boxed{\text{YES}}$-set, which we usually don't explicitly list.

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A computing problem is a $\boxed{\text{YES}}$-set, a set of *finite* binary strings.

Computing Problems Are Languages

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Solving the problem

Give a "procedure" to tell if a general input w is in the language ($\boxed{\text{YES}}$-set).

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$\mathcal{L}_{\text{unary}}$	$\{\varepsilon, 1, 11, 111, 1111, \ldots\} = \{1^{\bullet n} \mid n \geq 0\}$	$\leftarrow$ strings of 1s
$\mathcal{L}_{(01)^n}$	$\{\varepsilon, 01, 0101, 010101, \ldots\} = \{(01)^{\bullet n} \mid n \geq 0\}$	
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Computing Problems Are Languages

Language: Set of finite binary strings.

Solving the problem

Give a "procedure" to tell if a general input w is in the language ($\boxed{\text{YES}}$-set).

Abstract, precise and general formulation of a computing problem.

	$\{\varepsilon, 1, 10, 01\}$	$\leftarrow$ finite language
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Describing a Language: String Patterns and Variables

An example where there is a clear pattern,

$$\mathcal{L} = \{\varepsilon, 01, 0101, 010101, \ldots\}.$$

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← even palindromes

Pop Quiz. Formally define $\mathcal{L}_{\text{add}} = \{0100, 011000, 001000, 00110000, 00010000, 0001100000, 01110000, 0011100000, 000111000000, \ldots\}$

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For more complicated patterns, we use regular expressions, e.g. the Unix/Linux command

ls FOCS$*$

(Lists everything that starts with FOCS ($*$ is the "wild-card").)

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Pop Quiz. Which of the strings $\{101110, 00111, 00100, 01100\}$ can you generate using $\{0,01\}^* \bullet \{1,10\}^*$?

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Hence $1110111 \in \{1,11\} \bullet \overline{\{0,01\}^*} \bullet (\{00\} \cup \{1\}^*)$

Pop Quiz Is there another way to generate 1110111?

Pop Quiz Yes or no: $11110010 \in \{1,11\} \bullet \overline{\{0,01\}^*} \bullet (\{00\} \cup \{1\}^*)$?

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2. Regular expression for all palindromes (strings which equal their reversal)?

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How to define complexity of a computing problem?

Complexity of a Computing Problem

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How do we test membership?

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difficult problem $\leftrightarrow$ "complex" $\boxed{\text{YES}}$-set $\leftrightarrow$ hard to test membership in $\boxed{\text{YES}}$-set

How do we test membership? That brings us to *Models Of Computing*.

Visual encoding of four (machine-level) instructions:

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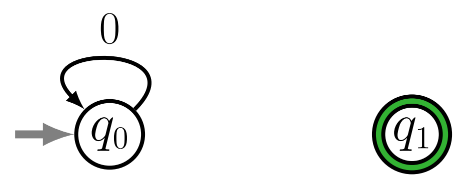
1 1 0 1



Visual encoding of four (machine-level) instructions:

1: In state q_0, when you process a 0, transition to state q_0.

Complexity of a Computing Problem

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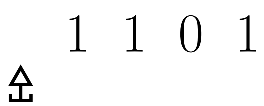

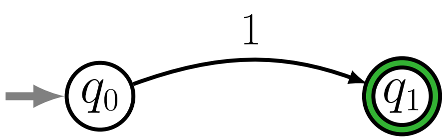

Visual encoding of four (machine-level) instructions:

1: In state q_0, when you process a 0, transition to state q_0.
2: In state q_0, when you process a 1, transition to state q_1.

Complexity of a Computing Problem

$$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$$ (strings ending in 1)

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How do we test membership? That brings us to *Models Of Computing*.

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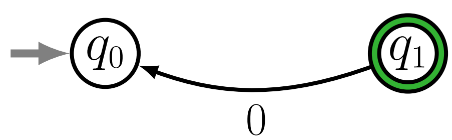

Visual encoding of four (machine-level) instructions:

1: In state q_0, when you process a 0, transition to state q_0.
2: In state q_0, when you process a 1, transition to state q_1.
3: In state q_1, when you process a 0, transition to state q_0.

Complexity of a Computing Problem

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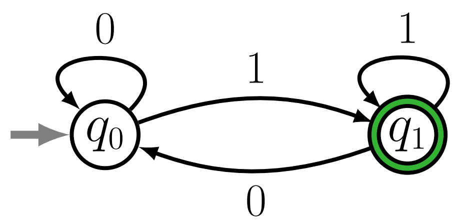

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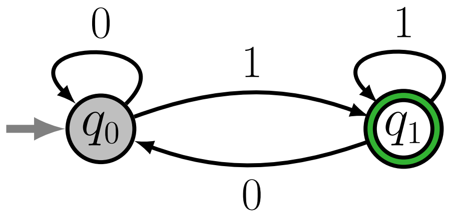
"Easy" to implement as a mechanical device.

A Simple Computing Machine (DFA)

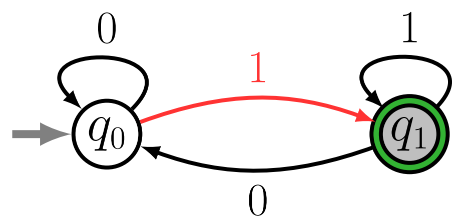
1 1 0 1



(current state in gray)

A Simple Computing Machine (DFA)

1 1 0 1



(current state in gray)

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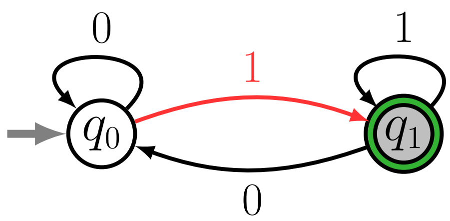
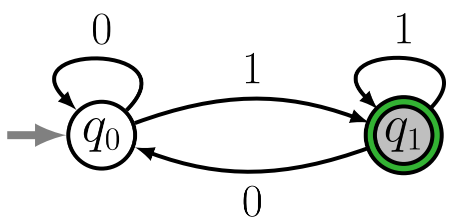
1 1 0 1



(current state in gray)

1 1 0 1



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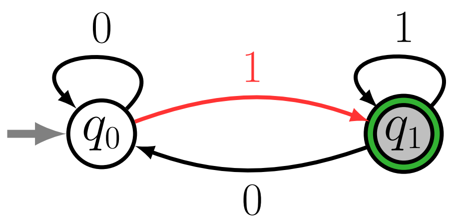
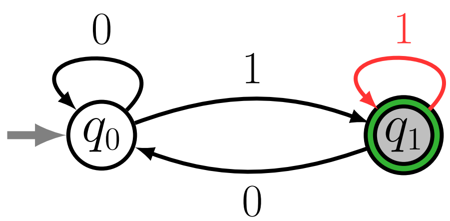


(current state in gray)

1 1 0 1



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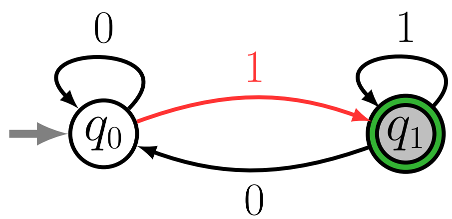

(current state in gray)

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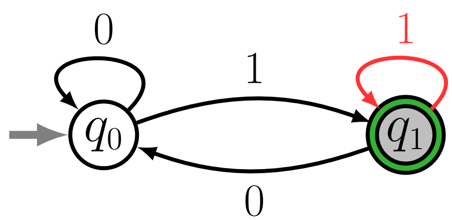
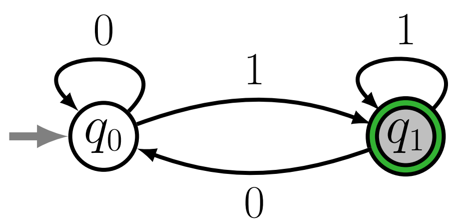
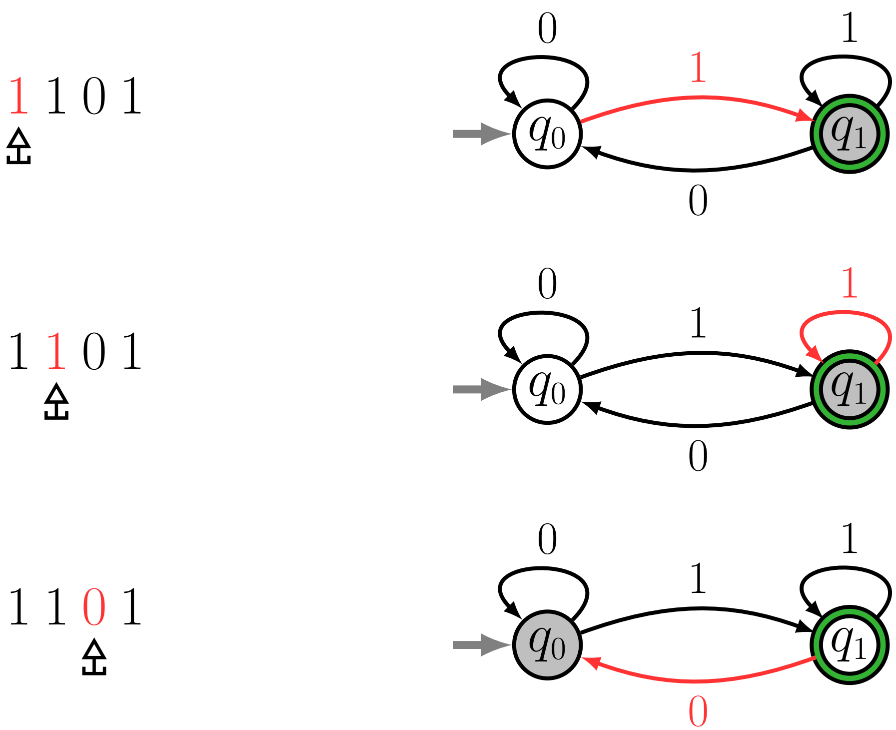


1 1 0 1



A Simple Computing Machine (DFA)



(current state in gray)

A Simple Computing Machine (DFA)

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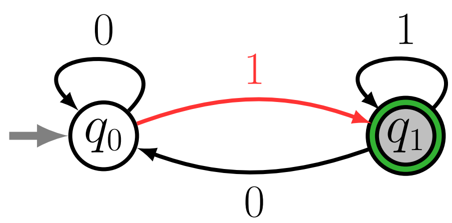
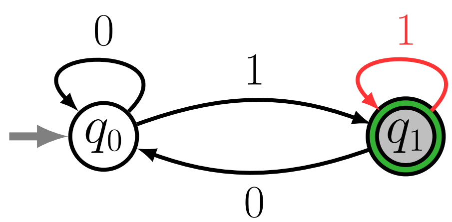
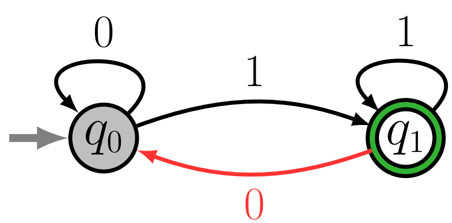


(current state in gray)

1 1 0 1



1 1 0 1



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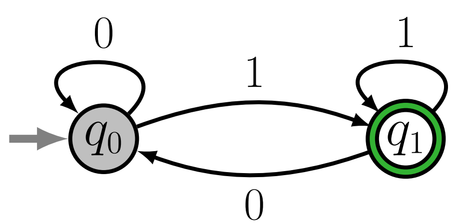

A Simple Computing Machine (DFA)

(current state in gray)

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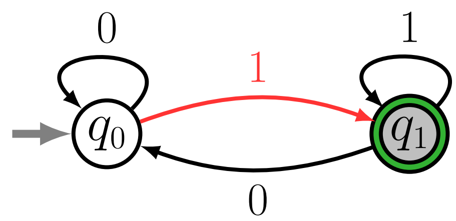

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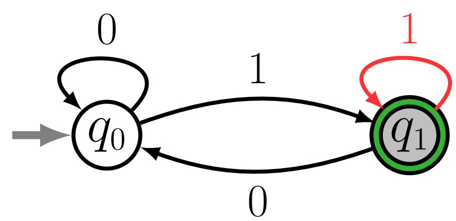

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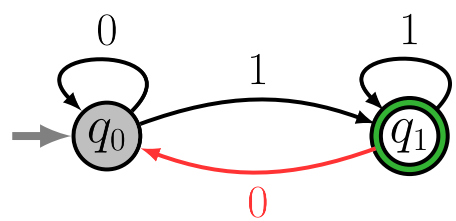

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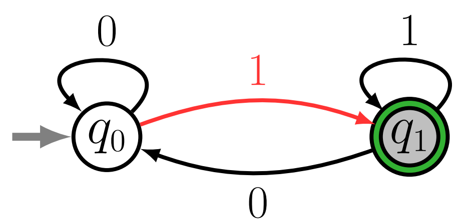

A Simple Computing Machine (DFA)

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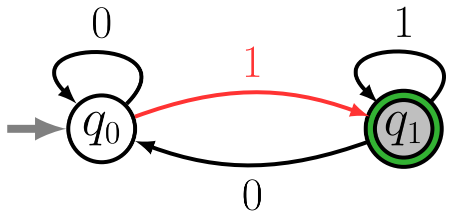
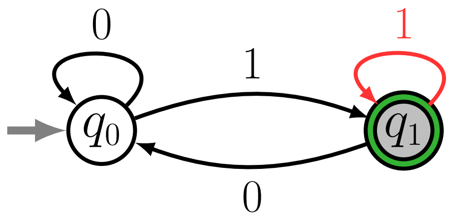


1 1 0 1



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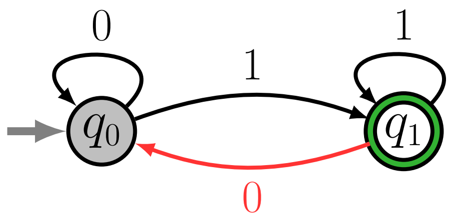

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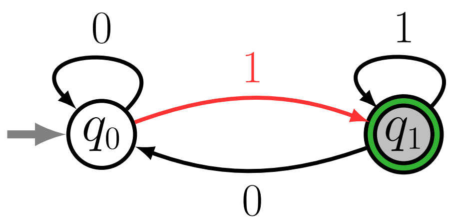
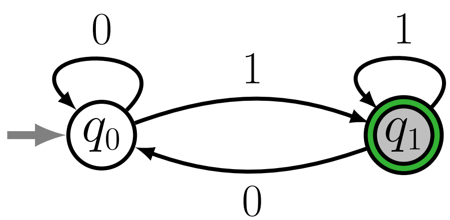


1 1 0 1



A Simple Computing Machine (DFA)

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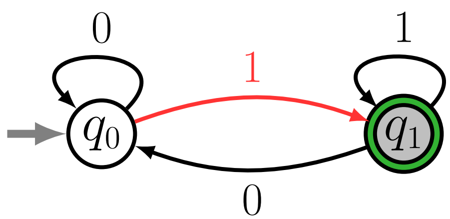

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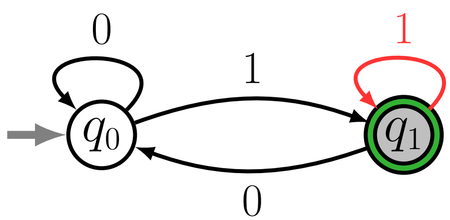

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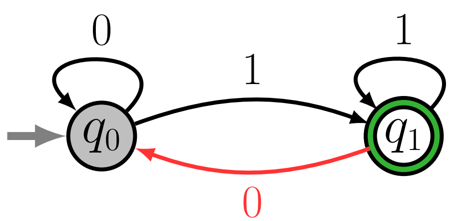

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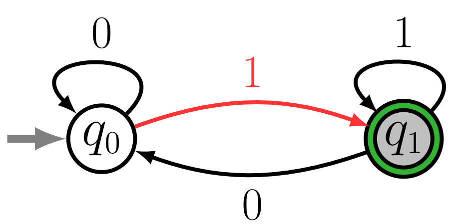

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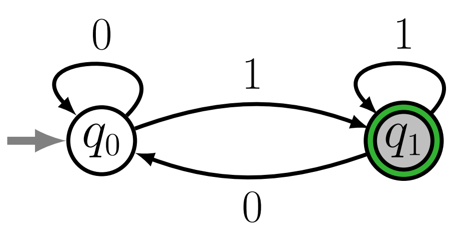


$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, \ldots\}$

Strings in $\mathcal{L}_{\text{push}}$ end in the "accepting" state q_1.

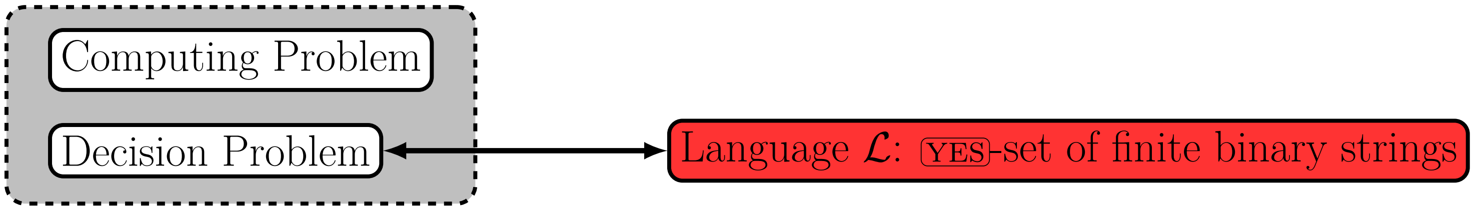
Strings not in $\mathcal{L}_{\text{push}}$ do not.

Computing Problems and Their Difficulty

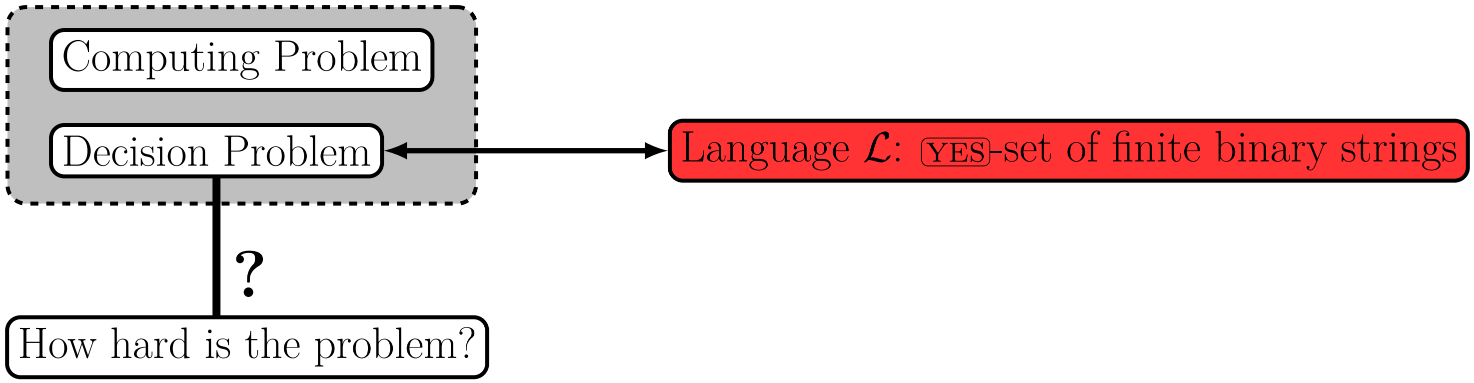
Computing Problem

Decision Problem

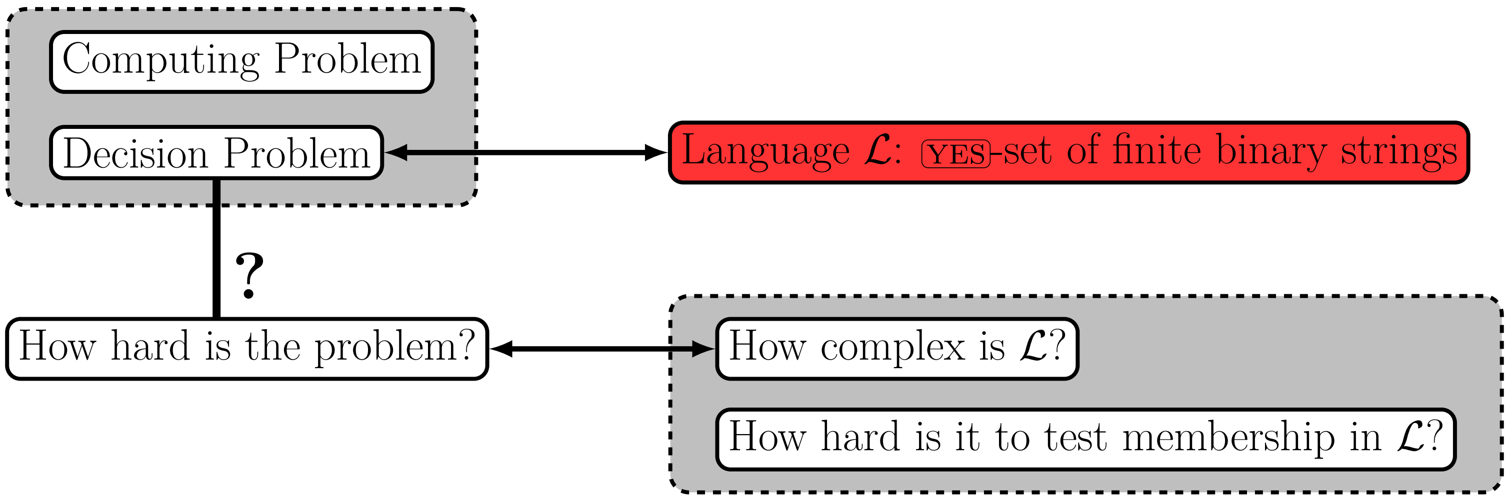
Computing Problems and Their Difficulty



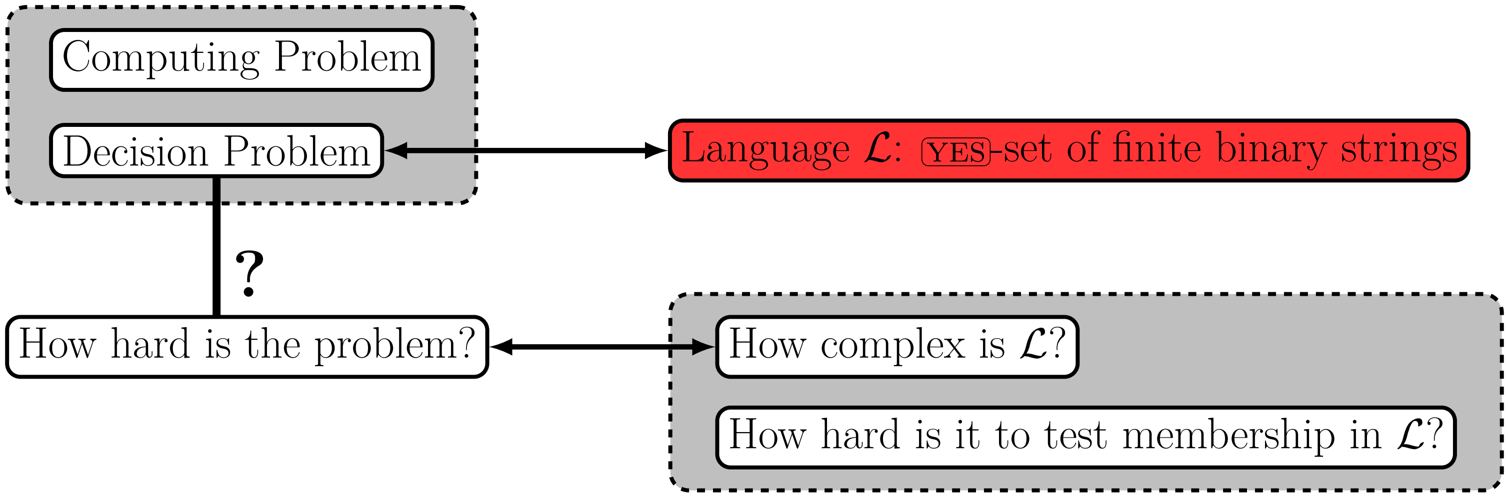
Computing Problems and Their Difficulty



Computing Problems and Their Difficulty



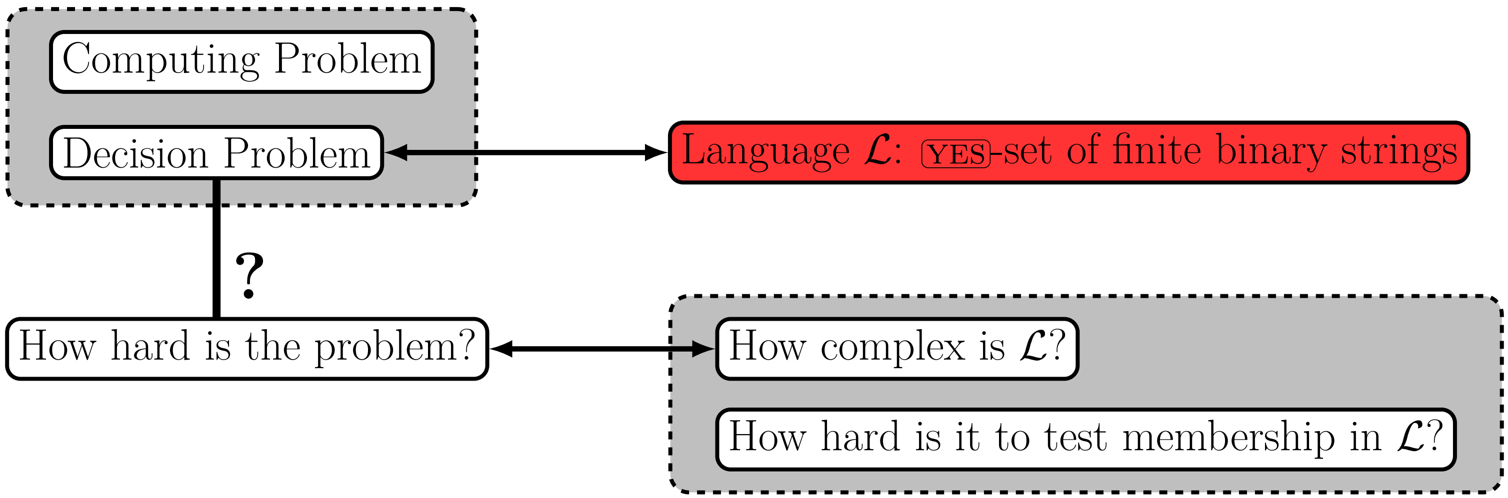
Computing Problems and Their Difficulty



A problem can be harder in two ways.

1. The problem needs more resources. For example, the problem can be solved with a similar machine to ours, except with more states.

Computing Problems and Their Difficulty



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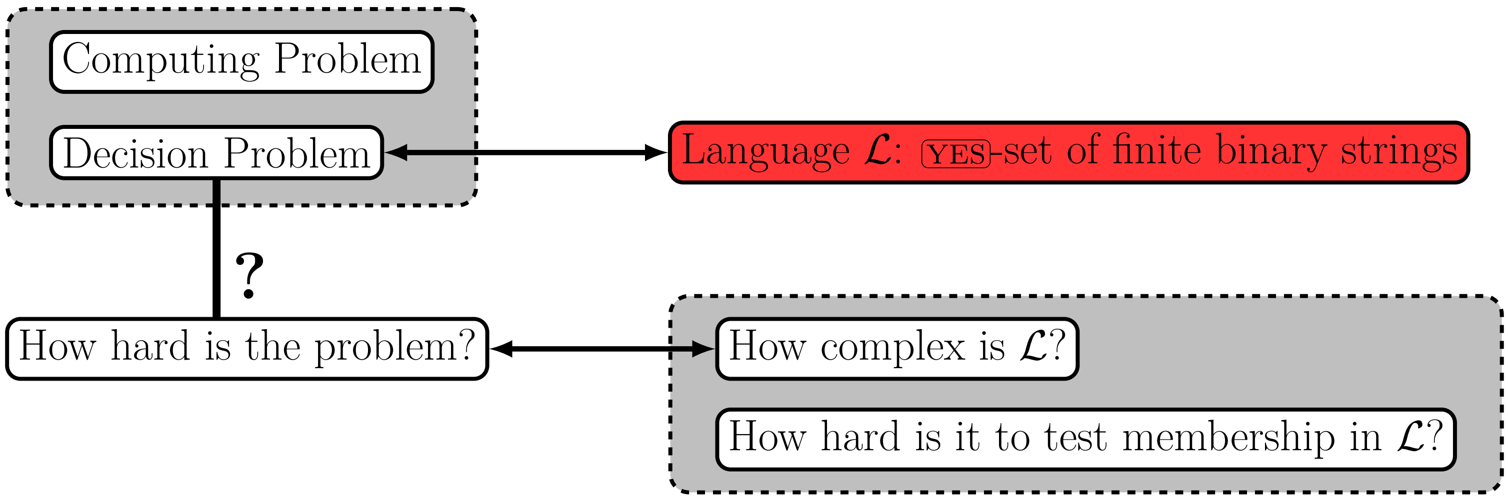
2. The problem needs a different *kind* of computing machine, with superior capabilities.

Computing Problems and Their Difficulty



A problem can be harder in two ways.

1. The problem needs more resources. For example, the problem can be solved with a similar machine to ours, except with more states.
2. The problem needs a different *kind* of computing machine, with superior capabilities.

The first type of "harder" is the focus of a follow-on algorithms course.

We focus on what *can and can't be solved* on a particular kind of machine.