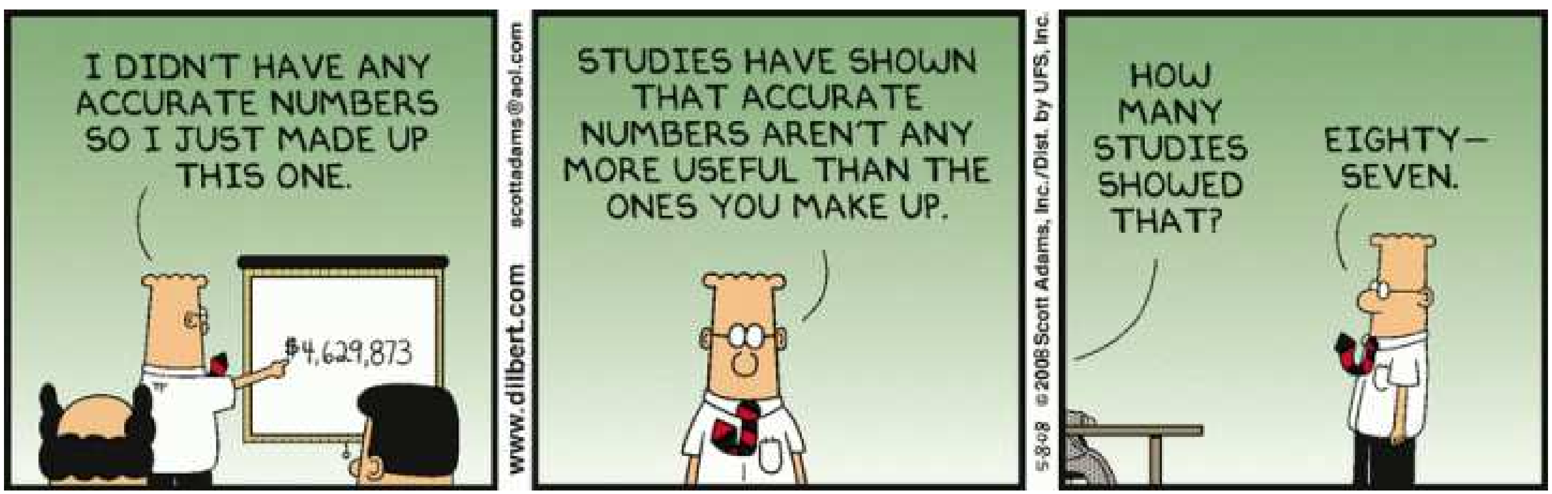
Foundations of Computer Science
Lecture 20

Expected Value of a Sum

Linearity of Expectation
Iterated Expectation
Build-Up Expectation
Sum of Indicators



Last Time

1. Sample average and expected value.
2. Definition of Mathematical expectation.
3. Examples: Sum of dice; Bernoulli; Uniform; Binomial; waiting time;
4. Conditional expectation.
5. Law of Total Expectation.

Today: Expected Value of a Sum

1. Expected value of a sum.
   - Sum of dice.
   - Binomial.
   - Waiting time.
   - Coupon collecting.
2. Iterated expectation.
3. Build-up expectation.
4. Expected value of a product.
5. Sum of indicators.

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Proof. $$\mathbb{E}[\mathbf{Z}] \;=\; \sum_{\omega\in\Omega}\big(a_1\mathbf{X}_1(\omega) + a_2\mathbf{X}_2(\omega) + \cdots + a_k\mathbf{X}_k(\omega)\big)\cdot P(\omega)$$

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■

1. **Summation can be taken inside or pulled outside an expectation.**
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$$\mathbb{E}\left[\sum_{i=1}^{k} a_i\mathbf{X}_i\right] \;=\; \sum_{i=1}^{k} a_i\,\mathbb{E}[\mathbf{X}_i]$$

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MUCH faster to observe that $\mathbf{X}$ is a sum,

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 + \mathbf{X}_4,$$

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$$= 4 \times 3\tfrac{1}{2} = 14.$$

$\leftarrow$in general $n \times 3\frac{1}{2}$

Exercise. Compute the full PDF for the sum of 4 dice and expected value from the PDF.

Expected Number of Successes in n Coin Tosses

$\mathbf{X}$ is the number of successes in n trials with success probability p per trial,

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$$\begin{aligned}
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&= \underset{p}{\mathbb{E}[\mathbf{X}_1]} + \underset{p}{\mathbb{E}[\mathbf{X}_2]} + \cdots + \underset{p}{\mathbb{E}[\mathbf{X}_n]} \\
&= n \times p.
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Example. If you are waiting for 3 boys, you have to wait 3-times as long as for 1 boy.

Exercise. Compute the expected *square* of the waiting time.

Coupon Collecting: Collecting the Flags

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Linearity of expectation:

$$\mathbb{E}[\mathbf{X}] = n(\tfrac{1}{n} + \tfrac{1}{n-1} + \tfrac{1}{n-2} + \cdots + \tfrac{1}{1}) = nH_n \approx n(\ln n + 0.577).$$

$n = 169 \rightarrow$ you expect to buy about 965 packs of gum. Lots of chewing!

Example. Cereal box contains 1-of-5 cartoon characters. Collect all to get \$2 rebate.

Expect to buy about 12 cereal boxes. If a cereal box costs \$5, that's a whopping $3\frac{1}{3}\%$ discount.

Iterated Expectation

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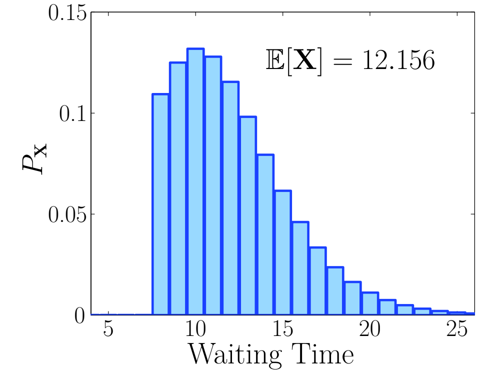
Exercise. Justify this computation using total expectation with 6 cases:

$$\mathbb{E}[\mathbf{X}_2] = \mathbb{E}[\mathbf{X}_2 \mid \mathbf{X}_1 = 1] \cdot \mathbb{P}[\mathbf{X}_1 = 1] + \mathbb{E}[\mathbf{X}_2 \mid \mathbf{X}_1 = 2] \cdot \mathbb{P}[\mathbf{X}_1 = 2] + \cdots + \mathbb{E}[\mathbf{X}_2 \mid \mathbf{X}_1 = 6] \cdot \mathbb{P}[\mathbf{X}_1 = 6].$$

Build-Up Expectation: Waiting for 2 Boys and 6 Girls

$$W(k, \ell) = \mathbb{E}[\text{waiting time to } k \text{ boys and } \ell \text{ girls}].$$



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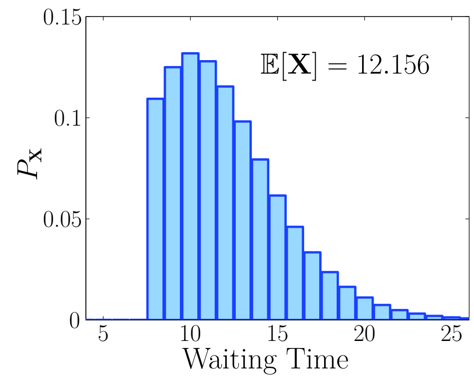
The first child is either a boy or girl, so by total expectation,

$$W(k,l) = \underbrace{\mathbb{E}[\text{waiting time} \mid \text{boy}]}_{1+W(k-1,\ell)} \times \underbrace{\mathbb{P}[\text{boy}]}_{p} + \underbrace{\mathbb{E}[\text{waiting time} \mid \text{girl}]}_{1+W(k,\ell-1)} \times \underbrace{\mathbb{P}[\text{girl}]}_{1-p}$$



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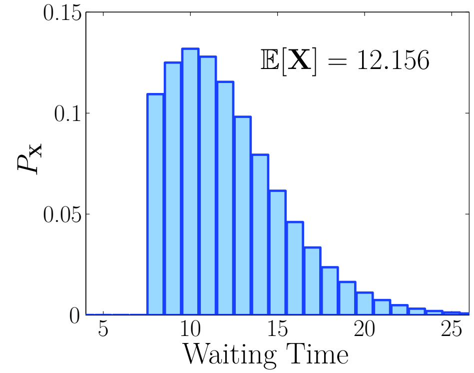

Build-Up Expectation: Waiting for 2 Boys and 6 Girls

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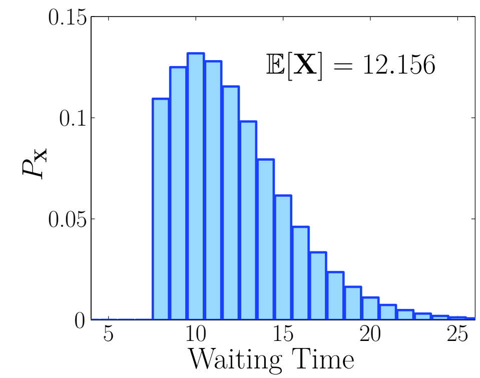

$W(k,\ell)$		ℓ: 0	1	2	3	4	5	6	7	$\cdots$
	0	0	2	4	6	8	10	12	14	$\cdots$
k	1	2								
	2	4								
	$\vdots$	$\vdots$								

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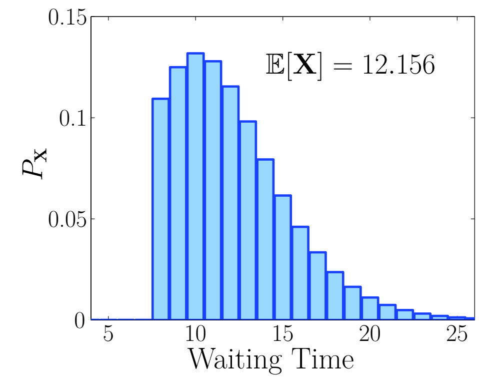

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k	0	0	2	4	6	8	10	12	14	$\cdots$
	1	$2 \xrightarrow{\times(1-p)}$	$\downarrow \times p$ $+1$ 3							
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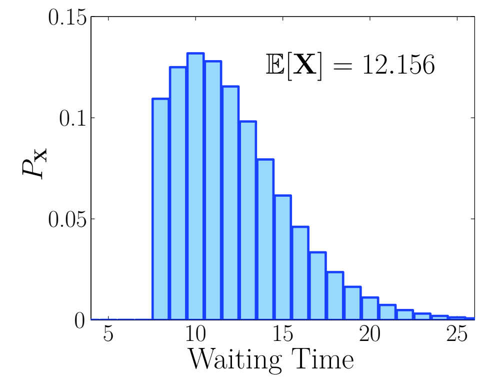

$W(k,\ell)$		ℓ = 0	1	2	3	4	5	6	7	$\cdots$
	0	0	2	4	6	8	10	12	14	$\cdots$
k	1	$2 \xrightarrow[\times(1-p)]{+1}$	$\overset{\times p}{3}$	4.5	6.25	8.13	10.06	12.03	14.02	$\cdots$
	2	4								
	$\vdots$	$\vdots$								

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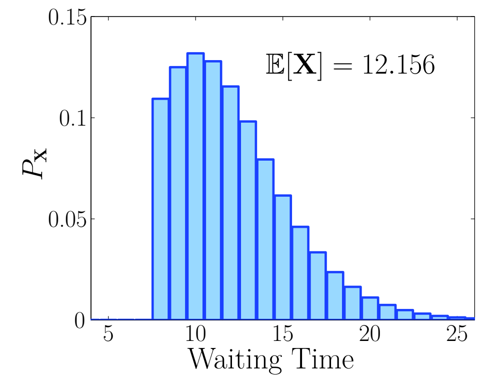

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k	1	2 $\xrightarrow[\times(1-p)]{}$	$\downarrow \times p$ $+1$ 3	4.5	6.25	8.13	10.06	12.03	14.02	$\cdots$
	2	4	4.5	5.5	6.88	8.5	10.28	12.16	14.09	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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$\mathbf{X}_1$ and $\mathbf{X}_2$ are independent die rolls:

Die 2 Value						
⚅	6	12	18	24	30	36
⚄	5	10	15	20	25	30
⚃	4	8	12	16	20	24
⚂	3	6	9	12	15	18
⚁	2	4	6	8	10	12
⚀	1	2	3	4	5	6
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Die 1 Value

Expected Value of a Product

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Expected value of a product XY.

1. **In general, the expected product is not a product of expectations.**
2. **For independent random variables , it is:** $\mathbb{E}[\mathbf{XY}] = \mathbb{E}[\mathbf{X}]\times\mathbb{E}[\mathbf{Y}]$.

Sum of Indicators: Successes in a Random Assignment

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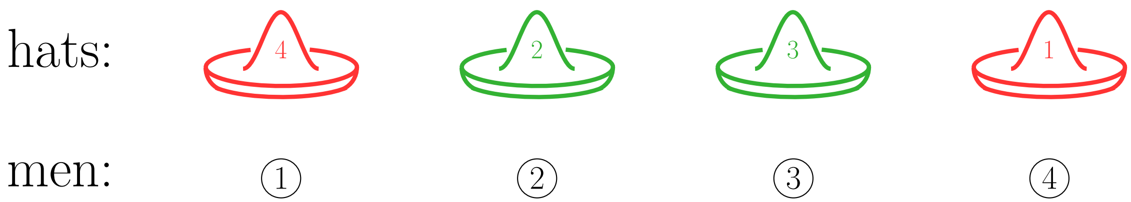

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hats:

men: 1 2 3 4

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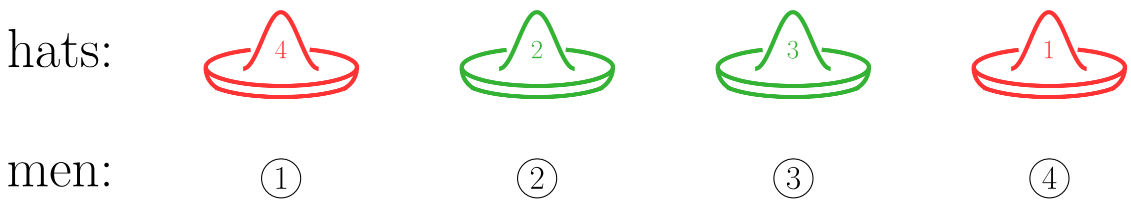

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men: ① ② ③ ④

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$$\mathbb{E}[\mathbf{X}] \;=\; \mathbb{E}[\mathbf{X}_1] + \mathbb{E}[\mathbf{X}_2] + \mathbb{E}[\mathbf{X}_4] + \mathbb{E}[\mathbf{X}_4] \;=\; \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} \;=\; 4 \times \tfrac{1}{4} \;=\; 1.$$

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Exercise. What about if there are n people?
Interesting Example (see text). Apply sum of indicators to breaking of records.
Instructive Exercise. Compute the PDF of $\mathbf{X}$ and the expectation from the PDF.