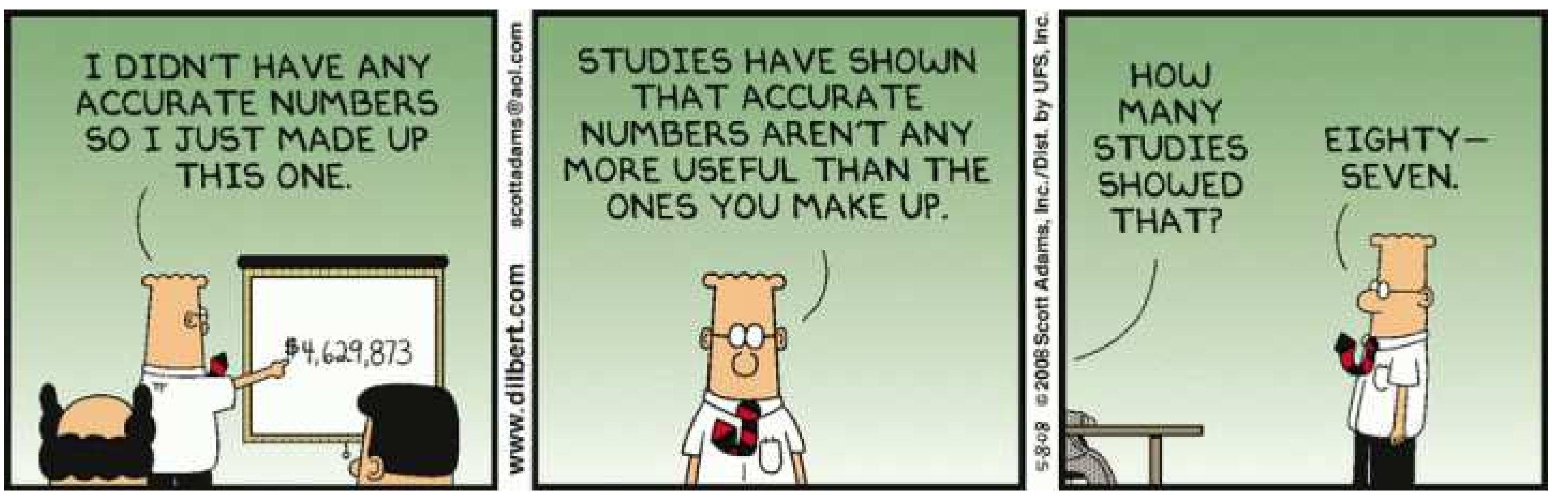
Foundations of Computer Science
Lecture 20

Expected Value of a Sum

Linearity of Expectation
Iterated Expectation
Build-Up Expectation
Sum of Indicators



Last Time

1. Sample average and expected value.
2. Definition of Mathematical expectation.
3. Examples: Sum of dice; Bernoulli; Uniform; Binomial; waiting time;
4. Conditional expectation.
5. Law of Total Expectation.

Today: Expected Value of a Sum

1. Expected value of a sum.
   - Sum of dice.
   - Binomial.
   - Waiting time.
   - Coupon collecting.
2. Iterated expectation.
3. Build-up expectation.
4. Expected value of a product.
5. Sum of indicators.

Expected Value of a Sum

You expect to win twice as much from two lottery tickets as from one.

The expected value of a sum is a sum of the expected values.

Theorem (Linearity of Expectation). Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$ be random variables and let $\mathbf{Z} = a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + \cdots + a_k\mathbf{X}_k$ be a *linear* combination of the $\mathbf{X}_i$. Then,

$$\mathbb{E}[\mathbf{Z}] = \mathbb{E}[a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + \cdots + a_k\mathbf{X}_k] = a_1\,\mathbb{E}[\mathbf{X}_1] + a_2\,\mathbb{E}[\mathbf{X}_2] + \cdots + a_k\,\mathbb{E}[\mathbf{X}_k].$$

Proof.

$$\begin{aligned}
\mathbb{E}[\mathbf{Z}] &= \sum_{\omega\in\Omega}\Big(a_1\mathbf{X}_1(\omega) + a_2\mathbf{X}_2(\omega) + \cdots + a_k\mathbf{X}_k(\omega)\Big)\cdot P(\omega) \\
&= a_1\sum_{\omega\in\Omega}\mathbf{X}_1(\omega)\cdot P(\omega) + a_2\sum_{\omega\in\Omega}\mathbf{X}_2(\omega)\cdot P(\omega) + \cdots + a_k\sum_{\omega\in\Omega}\mathbf{X}_k(\omega)\cdot P(\omega) \\
&= a_1\,\mathbb{E}[\mathbf{X}_1] + a_2\,\mathbb{E}[\mathbf{X}_2] + \cdots + a_k\,\mathbb{E}[\mathbf{X}_k].
\end{aligned}$$

■

1. **Summation can be taken inside or pulled outside an expectation.**
2. **Constants can be taken inside or pulled outside an expectation.**

$$\mathbb{E}\left[\sum_{i=1}^{k} a_i\mathbf{X}_i\right] = \sum_{i=1}^{k} a_i\,\mathbb{E}[\mathbf{X}_i]$$

Sum of Dice

Let $\mathbf{X}$ be the sum of 4 fair dice, what is $\mathbb{E}[\mathbf{X}]$?

sum	4	5	6	7	$\cdots$	24
$\mathbb{P}[\text{sum}]$	$\frac{1}{1296}$	$\frac{4}{1296}$	$\frac{10}{1296}$	?	$\cdots$	$\frac{1}{1296}$

$\rightarrow$ $\mathbb{E}[\mathbf{X}] = 4 \times \frac{1}{1296} + 5 \times \frac{4}{1296} + \cdots$

MUCH faster to observe that $\mathbf{X}$ is a sum,

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 + \mathbf{X}_4,$$

where $\mathbf{X}_i$ is the value rolled by die i and

$$\mathbb{E}[\mathbf{X}_i] = 3\tfrac{1}{2}.$$

Linearity of expectation:

$$\begin{aligned}
\mathbb{E}[\mathbf{X}] &= \mathbb{E}[\mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 + \mathbf{X}_4] = \underbrace{\mathbb{E}[\mathbf{X}_1]}_{3\frac{1}{2}} + \underbrace{\mathbb{E}[\mathbf{X}_2]}_{3\frac{1}{2}} + \underbrace{\mathbb{E}[\mathbf{X}_3]}_{3\frac{1}{2}} + \underbrace{\mathbb{E}[\mathbf{X}_4]}_{3\frac{1}{2}} \\
&= 4 \times 3\tfrac{1}{2} = 14.
\end{aligned}$$

$\leftarrow$in general $n \times 3\frac{1}{2}$

Exercise. Compute the full PDF for the sum of 4 dice and expected value from the PDF.

Expected Number of Successes in n Coin Tosses

$\mathbf{X}$ is the number of successes in n trials with success probability p per trial,

$$\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n$$

Each $\mathbf{X}_i$ is a Bernoulli and

$$\mathbb{E}[\mathbf{X}_i] = p.$$

Linearity of expectation,

$$\begin{aligned} \mathbb{E}[\mathbf{X}] &= \mathbb{E}[\mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n] \\ &= \underset{p}{\mathbb{E}[\mathbf{X}_1]} + \underset{p}{\mathbb{E}[\mathbf{X}_2]} + \cdots + \underset{p}{\mathbb{E}[\mathbf{X}_n]} \\ &= n \times p. \end{aligned}$$

Expected Waiting Time to n Successes

$\mathbf{X}$ is the waiting time for n successes with success probability p.

$$\begin{aligned}\mathbf{X} &= \underbrace{\text{wait to 1st}}_{\mathbf{X}_1} + \underbrace{\text{wait from 1st to 2nd}}_{\mathbf{X}_2} + \underbrace{\text{wait from 2nd to 3rd}}_{\mathbf{X}_3} + \cdots + \underbrace{\text{wait from } (n-1)\text{th to } n\text{th}}_{\mathbf{X}_n} \\ &= \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 + \cdots + \mathbf{X}_n.\end{aligned}$$

Each $\mathbf{X}_i$ is a waiting time to *one* success, so

$$\mathbb{E}[\mathbf{X}_i] = \frac{1}{p}.$$

Linearity of expectation:

$$\begin{aligned}\mathbb{E}[\mathbf{X}] &= \mathbb{E}[\mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n] \\ &= \underbrace{\mathbb{E}[\mathbf{X}_1]}_{1/p} + \underbrace{\mathbb{E}[\mathbf{X}_2]}_{1/p} + \cdots + \underbrace{\mathbb{E}[\mathbf{X}_n]}_{1/p} \\ &= n/p.\end{aligned}$$

Example. If you are waiting for 3 boys, you have to wait 3-times as long as for 1 boy.

Exercise. Compute the expected *square* of the waiting time.

Coupon Collecting: Collecting the Flags

A pack of gum comes with a flag (169 countries). $\mathbf{X}$ is the number of gum-purchases to get all the flags.

$$\begin{aligned}\mathbf{X} &= \underbrace{\text{wait to 1st}}_{\substack{\mathbf{X}_1\\ \uparrow \\ p_1=\frac{n}{n}}} + \underbrace{\text{wait from 1st to 2nd}}_{\substack{\mathbf{X}_1\\ \uparrow \\ p_2=\frac{n-1}{n}}} + \underbrace{\text{wait from 2nd to 3rd}}_{\substack{\mathbf{X}_1\\ \uparrow \\ p_3=\frac{n-2}{n}}} + \cdots + \underbrace{\text{wait from } (n-1)\text{th to } n\text{th}}_{\substack{\mathbf{X}_1\\ \uparrow \\ p_n=\frac{n-(n-1)}{n}}}\\ &= \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 + \cdots + \mathbf{X}_n.\end{aligned}$$

$\mathbb{E}[\mathbf{X}_i] = 1/p_i$,

$$\mathbb{E}[\mathbf{X}_1] = \tfrac{n}{n}, \quad \mathbb{E}[\mathbf{X}_2] = \tfrac{n}{n-1}, \quad \mathbb{E}[\mathbf{X}_3] = \tfrac{n}{n-2}, \quad \ldots, \quad \mathbb{E}[\mathbf{X}_n] = \tfrac{n}{n-(n-1)}.$$

Linearity of expectation:

$$\mathbb{E}[\mathbf{X}] = n(\tfrac{1}{n} + \tfrac{1}{n-1} + \tfrac{1}{n-2} + \cdots + \tfrac{1}{1}) = nH_n \approx n(\ln n + 0.577).$$

$n = 169 \rightarrow$ you expect to buy about 965 packs of gum. Lots of chewing!

Example. Cereal box contains 1-of-5 cartoon characters. Collect all to get \$2 rebate.

Expect to buy about 12 cereal boxes. If a cereal box costs \$5, that's a whopping $3\frac{1}{3}\%$ discount.

Iterated Expectation

Experiment. Roll a die and let $\mathbf{X}_1$ be the value. Now, roll a second die $\mathbf{X}_1$ times and let $\mathbf{X}_2$ be the sum of these $\mathbf{X}_1$ rolls of the second die.

An example outcome is $(4; 2, 1, 2, 6)$ with $\mathbf{X}_1 = 4$ and $\mathbf{X}_2 = 11$:

$$\mathbb{E}[\mathbf{X}_2 \mid \mathbf{X}_1] = \mathbf{X}_1 \times 3\tfrac{1}{2}.$$

The RHS is a *function* of $\mathbf{X}_1$, a random variable. Compute its expectation.

$$\begin{aligned}
\mathbb{E}[\mathbf{X}_2] &= \mathbb{E}_{\mathbf{X}_1}[\mathbb{E}[\mathbf{X}_2 \mid \mathbf{X}_1]] && \text{(another version of total expectation)} \\
&= \mathbb{E}[\mathbf{X}_1] \times 3\tfrac{1}{2} \\
&= 3\tfrac{1}{2} \times 3\tfrac{1}{2} = 12\tfrac{1}{4}.
\end{aligned}$$

Exercise. Justify this computation using total expectation with 6 cases:

$$\mathbb{E}[\mathbf{X}_2] = \mathbb{E}[\mathbf{X}_2 \mid \mathbf{X}_1 = 1] \cdot \mathbb{P}[\mathbf{X}_1 = 1] + \mathbb{E}[\mathbf{X}_2 \mid \mathbf{X}_1 = 2] \cdot \mathbb{P}[\mathbf{X}_1 = 2] + \cdots + \mathbb{E}[\mathbf{X}_2 \mid \mathbf{X}_1 = 6] \cdot \mathbb{P}[\mathbf{X}_1 = 6].$$

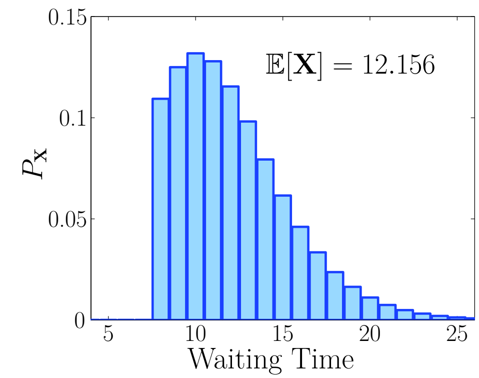
Build-Up Expectation: Waiting for 2 Boys and 6 Girls

$$W(k,\ell) = \mathbb{E}[\text{waiting time to } k \text{ boys and } \ell \text{ girls}].$$

The first child is either a boy or girl, so by total expectation,

$$\begin{aligned} W(k,l) &= \underbrace{\mathbb{E}[\text{waiting time} \mid \text{boy}]}_{1+W(k-1,\ell)} \times \underbrace{\mathbb{P}[\text{boy}]}_{p} + \underbrace{\mathbb{E}[\text{waiting time} \mid \text{girl}]}_{1+W(k,\ell-1)} \times \underbrace{\mathbb{P}[\text{girl}]}_{1-p} \\ &= 1 + pW(k-1,\ell) + (1-p)W(k,\ell-1). \end{aligned}$$

Base cases: $W(k,0) = k/p$ and $W(0,\ell) = \ell/(1-p)$



$W(k,\ell)$	$k \backslash \ell$	0	1	2	3	4	5	6	7	$\cdots$
	0	0	2 $\downarrow \times p$	4	6	8	10	12	14	$\cdots$
	1	2 $\xrightarrow{\times(1-p)}$	$\overset{+1}{3}$	4.5	6.25	8.13	10.06	12.03	14.02	$\cdots$
k	2	4	4.5	5.5	6.88	8.5	10.28	12.16	14.09	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Expected Value of a Product

$\mathbf{X}$ is a single die roll:

$$\mathbb{E}[\mathbf{X}^2] = \tfrac{1}{6}\cdot 1^2 + \tfrac{1}{6}\cdot 2^2 + \tfrac{1}{6}\cdot 3^2 + \tfrac{1}{6}\cdot 4^2 + \tfrac{1}{6}\cdot 5^2 + \tfrac{1}{6}\cdot 6^2 = \tfrac{91}{6} = 15\tfrac{1}{6}.$$

$$\mathbb{E}[\mathbf{X}^2] = \mathbb{E}[\mathbf{X}\times\mathbf{X}] = \mathbb{E}[\mathbf{X}]\times\mathbb{E}[\mathbf{X}] = (3\tfrac{1}{2})^2 = 12\tfrac{1}{4}. ✗$$

$\mathbf{X}_1$ and $\mathbf{X}_2$ are independent die rolls:

$$\begin{aligned}\mathbb{E}[\mathbf{X}_1\mathbf{X}_2] &= \tfrac{1}{36}(1+2+\cdots+6+2+4+\cdots+12+3+6+\cdots+18+\cdots+6+12+\cdots+36)\\ &= \tfrac{441}{36} = 12\tfrac{1}{4}.\end{aligned}$$

Die 2 Value						
6	6	12	18	24	30	36
5	5	10	15	20	25	30
4	4	8	12	16	20	24
3	3	6	9	12	15	18
2	2	4	6	8	10	12
1	1	2	3	4	5	6
Die 1 Value	1	2	3	4	5	6

$$\mathbb{E}[\mathbf{X}_1\mathbf{X}_2] = \mathbb{E}[\mathbf{X}_1]\times\mathbb{E}[\mathbf{X}_2] = (3\tfrac{1}{2})^2 = 12\tfrac{1}{4}. ✓$$

Expected value of a product XY.

1. **In general, the expected product is <u>not</u> a product of expectations.**
2. **For <u>independent</u> random variables , it is:** $\mathbb{E}[\mathbf{XY}] = \mathbb{E}[\mathbf{X}]\times\mathbb{E}[\mathbf{Y}]$.

Sum of Indicators: Successes in a Random Assignment

$\mathbf{X}$ is the number of correct hats when 4 hats randomly land on 4 heads.

hats:

men: ① ② ③ ④

$\mathbf{X}_1\!=\!0$ $\quad$ $\mathbf{X}_2\!=\!1$ $\quad$ $\mathbf{X}_3\!=\!1$ $\quad$ $\mathbf{X}_4\!=\!0$

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 + \mathbf{X}_4 = 2$$

$\mathbf{X}_i$ are Bernoulli with $\mathbb{P}[\mathbf{X}_i = 1] = \frac{1}{4}$. Linearity of expectation:

$$\mathbb{E}[\mathbf{X}] \;=\; \mathbb{E}[\mathbf{X}_1] + \mathbb{E}[\mathbf{X}_2] + \mathbb{E}[\mathbf{X}_4] + \mathbb{E}[\mathbf{X}_4] \;=\; \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} \;=\; 4 \times \tfrac{1}{4} \;=\; 1.$$

Exercise. What about if there are n people?
Interesting Example (see text). Apply sum of indicators to breaking of records.
Instructive Exercise. Compute the PDF of $\mathbf{X}$ and the expectation from the PDF.