Foundations of Computer Science
Lecture 15
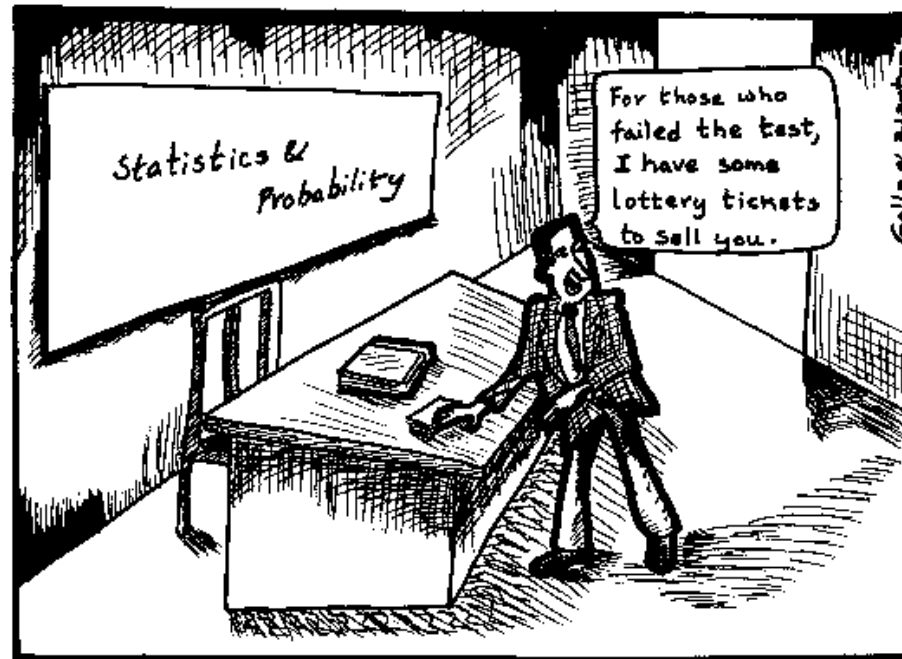
Probability

The probable is what usually happens – Aristotle

Last Time

> To count complex objects, construct a sequence of "instructions" that can be used to construct the object uniquely. The number of possible *sequences* of instructions equals the number of possible complex objects.

1. Counting
   - Sequences with and without repetition.
   - Subsets with and without repetition.
   - Sequences with specified numbers of each type of object: anagrams.

2. Inclusion-Exclusion (advanced technique).

3. Pigeonhole principle (simple but IMPORTANT technique).

Today: Probability

1 Computing probabilities.
- Outcome tree.
- Event of interest.
- Examples with dice.

2 Probability and sets.
- The probability space.

3 Uniform probability spaces.

4 Infinite probability spaces.

The Chance of Rain Tomorrow is 40%

What does the title mean? Either it will rain tomorrow or it won't.

The chances are 50% that a *fair* coin-flip will be H.

        Flip 100 times. Approximately 50 will be H   ← *frequentist* view.

1. You toss a *fair* coin 3 times. How many heads will you get?

2. You keep tossing a *fair* coin until you get a head. How many tosses will you make?

There's no answer. The outcome is uncertain. Probability handles such settings.

Birth of Mathematical Probability.

Antoine Gombaud,:	Should I bet even money on at least one 'double-6' in 24 rolls of two dice?
Chevalier de Méré	What about at least one 6 in 4 rolls of one die?
Blaise Pascal:	Interesting question. Let's bring *Pierre de Fermat* into the conversation.

        . . . a correspondence is ignited between these two mathematical giants
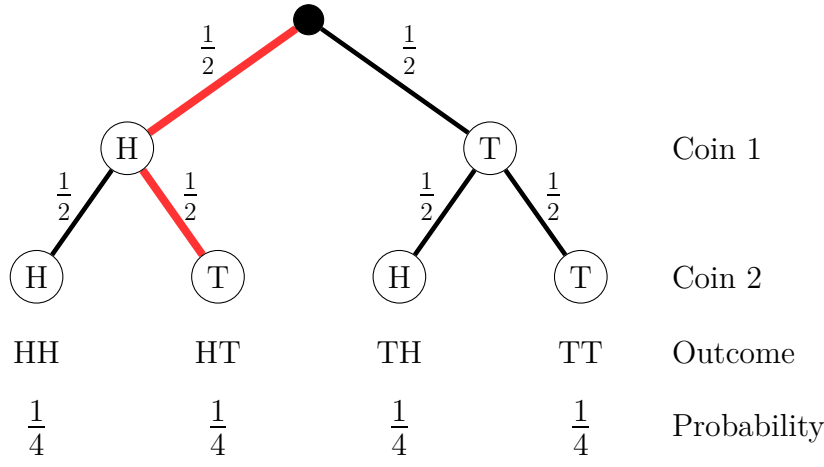
Toss Two Coins: You Win if the Coins Match (HH or TT)

① You are analyzing an "experiment" whose outcome is uncertain.

② **Outcomes.** Identify all *possible* outcomes using a *tree* of outcome *sequences*.

③ **Edge probabilities.** If one of k edges (options) from a vertex is chosen *randomly* then each edge has edge-probability $\frac{1}{k}$.

④ **Outcome-probability.** Multiply edge-probabilities to get outcome-probabilities.

H			T	Coin 1
H	T	H	T	Coin 2
HH	HT	TH	TT	Outcome
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	Probability

Event of Interest

Toss two coins: you win if the coins match (HH or TT)
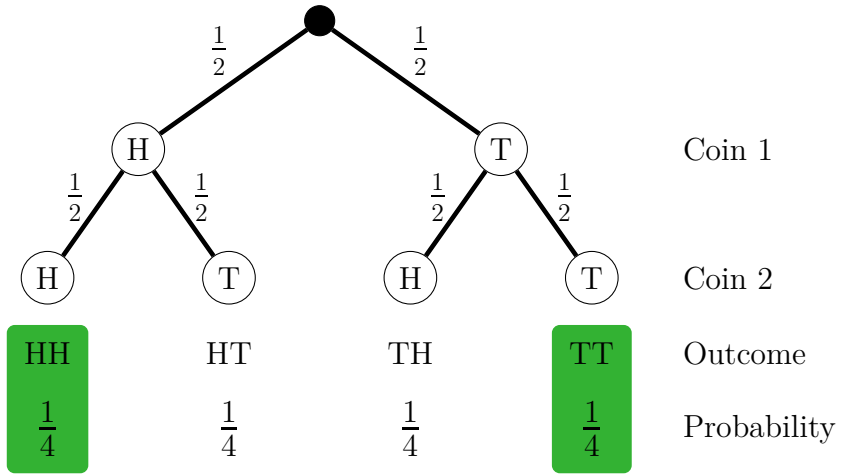
Question: When do you win?   **Event:** *Subset* of outcomes where you win.

⑤ Event of interest. Subset of the outcomes where you win.

⑥ Event-probability. Sum of its outcome-probabilities.

$$\text{event-probability} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Probability that you win is $\frac{1}{2}$, written as $\mathbb{P}[\text{"YouWin"}] = \frac{1}{2}$.

Go and do this experiment at home. Toss two coins 1000 times and see how often you win.

The Outcome-Tree Method

Become familiar with this 6-step process for analyzing a probabilistic experiment.

> **①** You are analyzing an experiment whose outcome is uncertain.
>
> **②** **Outcomes.** Identify *all possible* outcomes, the tree of *outcome sequences*.
>
> **③** **Edge-Probability.** Each edge in the outcome-tree gets a probability.
>
> **④** **Outcome-Probability.** Multiply edge-probabilities to get outcome-probabilities.
>
> **⑤** **Event of Interest $\mathcal{E}$.** Determine the subset of the outcomes you care about.
>
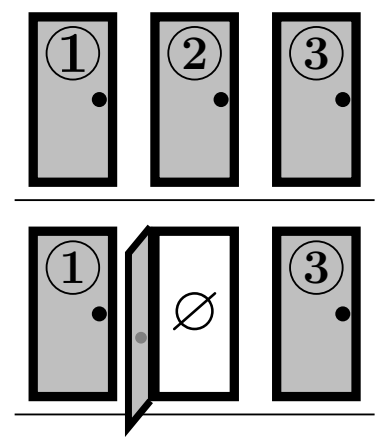> **⑥** **Event-Probability.** The sum of outcome-probabilities in the subset you care about.
> $$\mathbb{P}[\mathcal{E}] = \sum_{\text{outcomes } \omega \,\in\, \mathcal{E}} P(\omega).$$
>
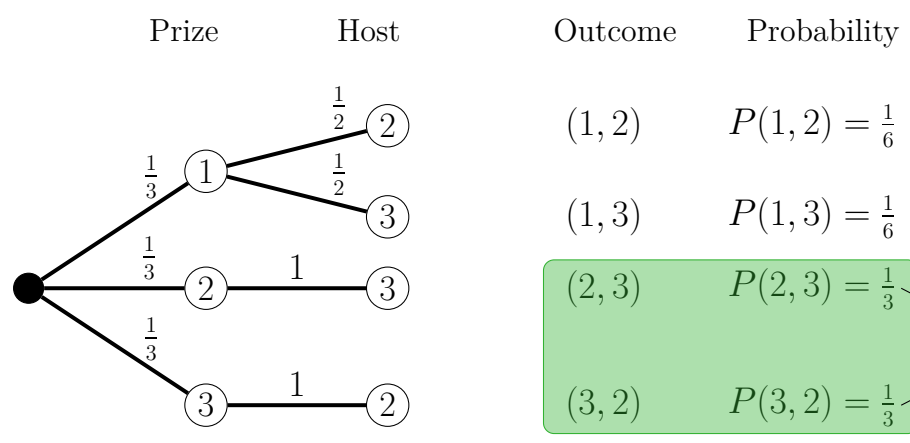> $\mathbb{P}[\mathcal{E}] \sim$ frequency an outcome you want occurs over many repeated experiments.

Pop Quiz. Roll two dice. Compute $\mathbb{P}[\text{first roll is less than the second}]$.

Let's Make a Deal: The Monty Hall Problem

1: Contestant at door 1.
2: Prize placed behind *random* door.
3: Monty opens <u>empty</u> door (*randomly* if there's an option).

- Outcome-tree and edge-probabilities.
- Outcome-probabilities.
- Event of interest: "WinBySwitching".
- Event probability.

Prize	Host	Outcome	Probability
		$(1,2)$	$P(1,2) = \frac{1}{6}$
		$(1,3)$	$P(1,3) = \frac{1}{6}$
		$(2,3)$	$P(2,3) = \frac{1}{3}$
		$(3,2)$	$P(3,2) = \frac{1}{3}$

$$\frac{1}{3} + \frac{1}{3} = \frac{2}{3} = \mathbb{P}[\text{"WinBySwitching"}]$$

Non-Transitive Dice

A: $\{ \boxed{\cdot\cdot}\ \boxed{\vdots}\ \boxed{\vdots\vdots}\ \boxed{\cdot\cdot}\ \boxed{\vdots}\ \boxed{\vdots\vdots} \}$ B: $\{ \boxed{\cdot}\ \boxed{\because}\ \boxed{\vdots\vdots}\ \boxed{\cdot}\ \boxed{\because}\ \boxed{\vdots\vdots} \}$ C: $\{ \boxed{\cdot}\ \boxed{\because}\ \boxed{\vdots}\ \boxed{\cdot}\ \boxed{\because}\ \boxed{\vdots} \}$
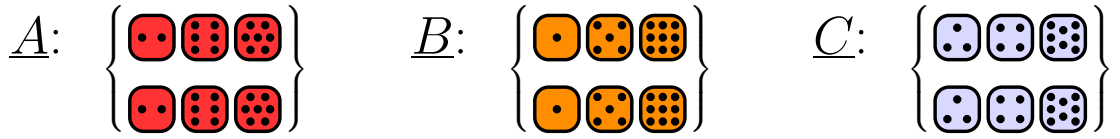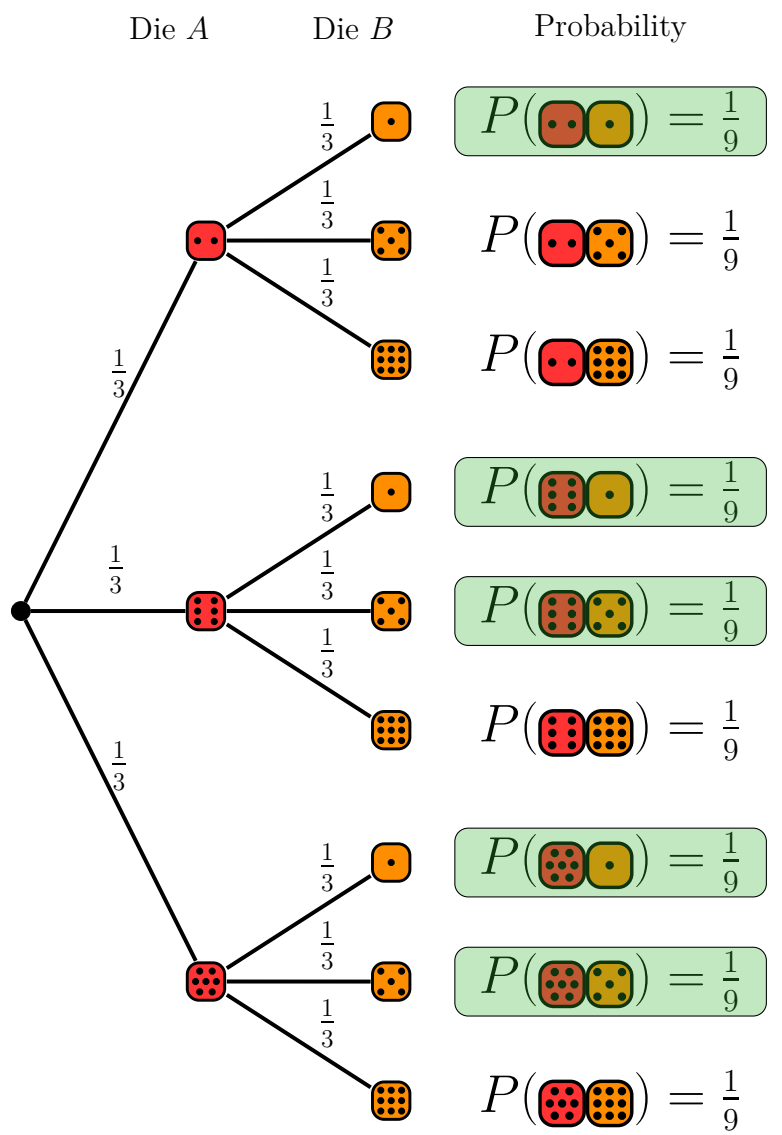
Your friend picks a die and then you pick a die.
E.g. friend picks B and then you pick A.

What is the probability that A beats B?

- Outcome-tree and outcome-probabilities.

- Uniform probabilities.

- Even of interest: outcomes where A wins.

- Number of outcomes where A wins: 5.

- $\mathbb{P}[A \text{ beats } B] = \frac{5}{9}$.

Conclusion: Die A beats Die B.

Die A Die B Probability

$P(\boxed{\cdot\cdot}\,\boxed{\cdot}) = \frac{1}{9}$

$P(\boxed{\cdot\cdot}\,\boxed{\because}) = \frac{1}{9}$

$P(\boxed{\cdot\cdot}\,\boxed{\vdots\vdots}) = \frac{1}{9}$

$P(\boxed{\vdots}\,\boxed{\cdot}) = \frac{1}{9}$

$P(\boxed{\vdots}\,\boxed{\because}) = \frac{1}{9}$

$P(\boxed{\vdots}\,\boxed{\vdots\vdots}) = \frac{1}{9}$

$P(\boxed{\vdots\vdots}\,\boxed{\cdot}) = \frac{1}{9}$

$P(\boxed{\vdots\vdots}\,\boxed{\because}) = \frac{1}{9}$

$P(\boxed{\vdots\vdots}\,\boxed{\vdots\vdots}) = \frac{1}{9}$

Pop Quiz. Compute $\mathbb{P}[B \text{ beats } C]$ and $\mathbb{P}[C \text{ beats } A]$ and show A beats B, B beats C and C beats A.

Probability and Sets: The Probability Space

> ① **Sample Space** $\Omega = \{\omega_1, \omega_2, \ldots\}$, set of *possible* outcomes.
> ② **Probability Function** $P(\cdot)$. Non-negative function $P(\omega)$, normalized to 1:
> $$0 \le P(\omega) \le 1 \qquad \text{and} \qquad \sum_{\omega \in \Omega} P(\omega) = 1.$$
>
> (Die A versus B) $\quad \Omega \;\Big|\; \{ \square, \square, \square, \square, \square, \square, \square, \square, \square \}$
>
> $\qquad\qquad\qquad P(\omega) \;\Big|\; \frac{1}{9} \quad \frac{1}{9} \quad \frac{1}{9} \quad \frac{1}{9} \quad \frac{1}{9} \quad \frac{1}{9} \quad \frac{1}{9} \quad \frac{1}{9} \quad \frac{1}{9}$

Events $\mathcal{E} \subseteq \Omega$ are subsets. Event probability $\mathbb{P}[\mathcal{E}]$ is the sum of outcome-probabilities.

$$\text{``}A > B\text{''} \qquad \mathcal{E}_1 \;=\; \{ \square, \square, \square, \square, \square \}$$

$$\text{``Sum} > 8\text{''} \qquad \mathcal{E}_2 \;=\; \{ \square, \square, \square, \square, \square \}$$

$$\text{``}B < 9\text{''} \qquad \mathcal{E}_3 \;=\; \{ \square, \square, \square, \square, \square, \square \}$$

Combining events using logical connectors corresponds to set operations:

$$\text{``}A > B\text{''} \vee \text{``Sum} > 8\text{''} \qquad \mathcal{E}_1 \cup \mathcal{E}_2 = \{ \square, \square, \square, \square, \square, \square, \square, \square \}$$

$$\text{``}A > B\text{''} \wedge \text{``Sum} > 8\text{''} \qquad \mathcal{E}_1 \cap \mathcal{E}_2 = \{ \square, \square \}$$

$$\neg(\text{``}A > B\text{''}) \qquad \overline{\mathcal{E}_1} = \{ \square, \square, \square, \square \}$$

$$\text{``}A > B\text{''} \to \text{``}B < 9\text{''} \qquad \mathcal{E}_1 \subseteq \mathcal{E}_3$$

Important: Exercise 15.10. Sum rule, complement, inclusion-exclusion, union, implication and intersection bounds.

Uniform Probability Space : Probability $\sim$ Size

$$P(\omega) = \frac{1}{|\Omega|} \qquad \mathbb{P}\left[\mathcal{E}\right] = \frac{|\mathcal{E}|}{|\Omega|} = \frac{\text{number of outcomes in } \mathcal{E}}{\text{number of possible outcomes in } \Omega}.$$

Toss a coin 3 times:

	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT	Outcome
	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	Probability
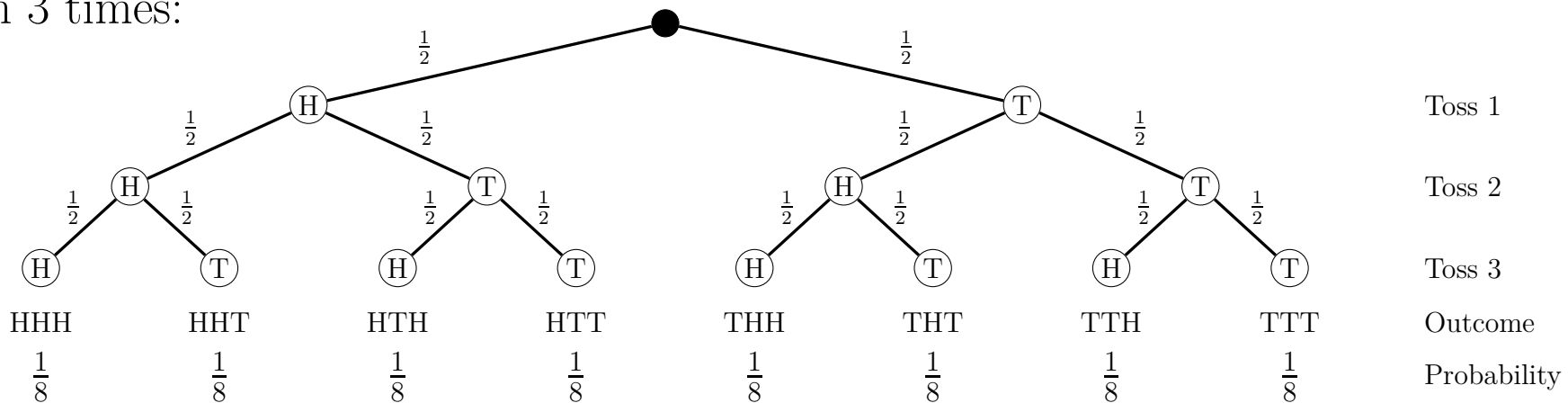
$$\mathbb{P}[\text{``2 heads''}] = \frac{\text{number of sequences with 2 heads}}{\text{number of possible sequences in } \Omega} = \binom{3}{2} \times \frac{1}{8} = \frac{3}{8}.$$

> **Practice: Exercise 15.10.**
> ① You roll a pair of regular dice. What is the probability that the sum is 9?
> ② You toss a fair coin ten times. What is the probability that you obtain 4 heads?
> ③ You roll die A ten times. Compute probabilities for: 4 sevens?   4 sevens AND 3 sixes?   4 sevens OR 3 sixes?

Poker: Probabilities of Full House and Flush

52 card deck has 4 suits ($\spadesuit$, $\heartsuit$, $\diamondsuit$, $\clubsuit$) and 13 ranks in a suit (A,K,Q,J,T,9,8,7,6,5,4,3,2).

Randomly deal 5-cards: each *set* of 5 cards is equally likely $\rightarrow$ uniform probability space.

$$\text{number of possible outcomes} = \binom{52}{5} \text{ possible hands.}$$

Full house: 3 cards of one rank and 2 of another. How many full-houses?
To construct a full house, specify ($\mathsf{rank}_3, \mathsf{suits}_3, \mathsf{rank}_2, \mathsf{suits}_2$). Product rule:

$$\text{\# full houses} = 13 \times \binom{4}{3} \times 12 \times \binom{4}{2} \quad \rightarrow \quad \mathbb{P}[\text{``FullHouse''}] = \frac{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}} \approx 0.00144;$$
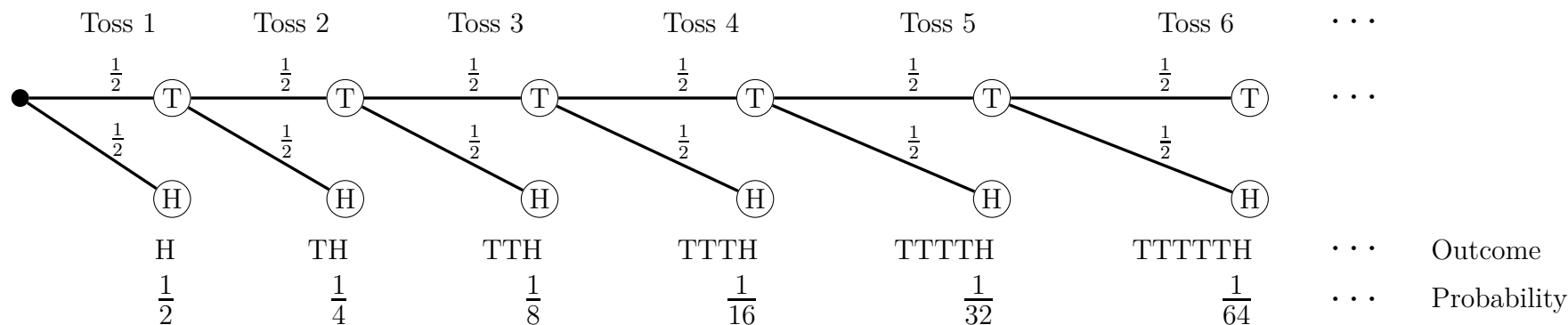
Flush: 5 cards of same suit. How many flushes?
To construct a flush, specify ($\mathsf{suit}, \mathsf{ranks}$). Product rule:

$$\text{\# flushes} = 4 \times \binom{13}{5} \quad \rightarrow \quad \mathbb{P}[\text{``Flush''}] = \frac{4 \times \binom{13}{5}}{\binom{52}{5}} \approx 0.00198;$$

Full house is rarer. That's why full house beats flush.

Toss a Coin Until Heads: Infinite Probability Space

Ω	H	TH	$T^{\bullet 2}H$	$T^{\bullet 3}H$	$T^{\bullet 4}H$	$T^{\bullet 5}H$	$\cdots$	$T^{\bullet i}H$	$\cdots$
$P(\omega)$	$\frac{1}{2}$	$(\frac{1}{2})^2$	$(\frac{1}{2})^3$	$(\frac{1}{2})^4$	$(\frac{1}{2})^5$	$(\frac{1}{2})^6$	$\cdots$	$(\frac{1}{2})^{i+1}$	$\cdots$
# Tosses	1	2	3	4	5	6	$\cdots$	$i+1$	$\cdots$

Sum of outcome probabilities:

$$\tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + (\tfrac{1}{2})^4 + \cdots = \sum_{i=1}^{\infty}(\tfrac{1}{2})^i = \frac{\tfrac{1}{2}}{1 - \tfrac{1}{2}} = 1. \ ✔$$

Game: First Person To Toss H Wins. **Always Go First**

$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Outcome:
H · TH · TTH · TTTH · TTTTH · TTTTTH · · · · Outcome

$\frac{1}{2}$ · $\frac{1}{4}$ · $\frac{1}{8}$ · $\frac{1}{16}$ · $\frac{1}{32}$ · $\frac{1}{64}$ · · · Probability
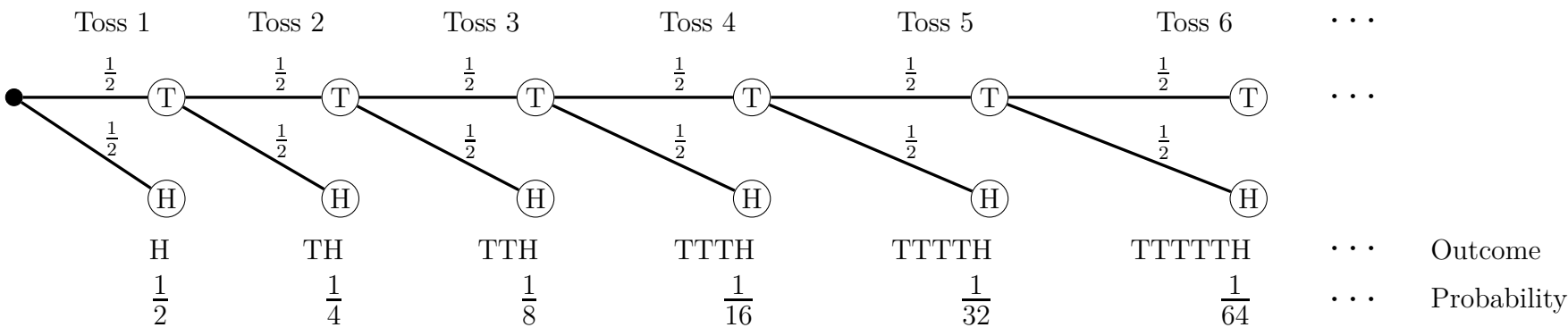
Ω	H	TH	$\text{T}^{\bullet 2}\text{H}$	$\text{T}^{\bullet 3}\text{H}$	$\text{T}^{\bullet 4}\text{H}$	$\text{T}^{\bullet 5}\text{H}$	$\cdots$	$\text{T}^{\bullet i}\text{H}$	$\cdots$
$P(\omega)$	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^4$	$\left(\frac{1}{2}\right)^5$	$\left(\frac{1}{2}\right)^6$	$\cdots$	$\left(\frac{1}{2}\right)^{i+1}$	$\cdots$

The event "YouWin" is $\mathcal{E} = \{\text{H}, \text{T}^{\bullet 2}\text{H}, \text{T}^{\bullet 4}\text{H}, \text{T}^{\bullet 6}\text{H}, \ldots\}$.

$$\mathbb{P}[\text{"YouWin"}] = \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^5 + \left(\tfrac{1}{2}\right)^7 + \cdots = \tfrac{1}{2}\sum_{i=0}^{\infty}\left(\tfrac{1}{4}\right)^i = \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \mathbf{\tfrac{2}{3}}.$$

Your odds improve by a factor of 2 if you go first (vs. second).