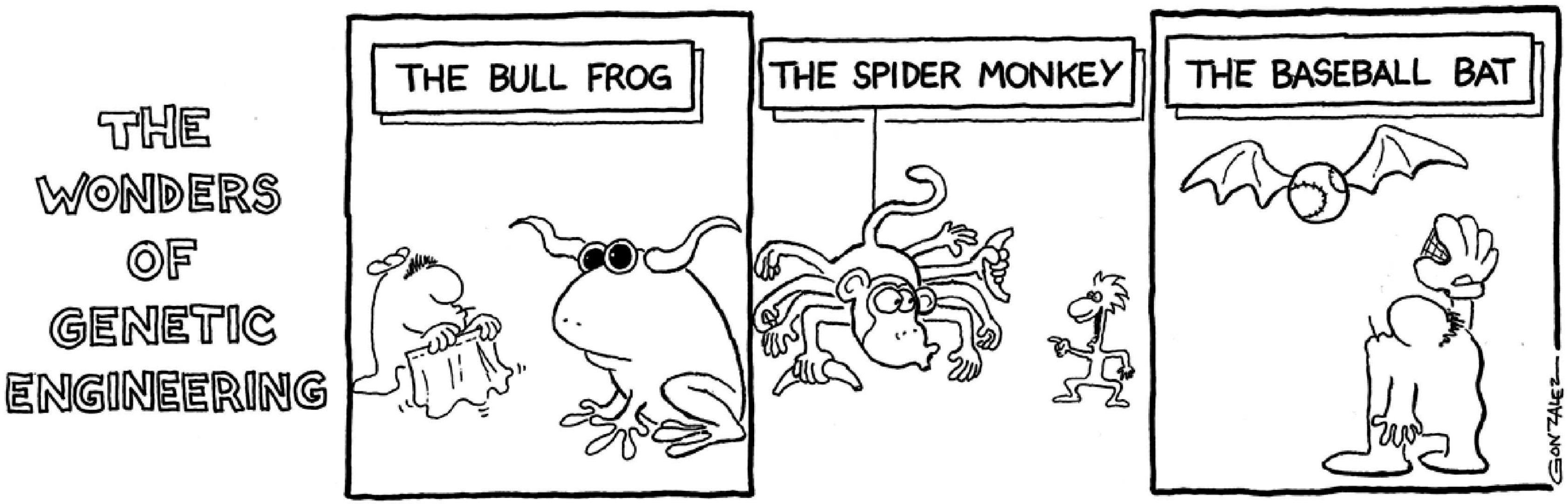
Foundations of Computer Science
Lecture 14

Advanced Counting

Sequences with Repetition
Union of Overlapping Sets: Inclusion-Exclusion
Pigeonhole Principle



Last Time

> To count complex objects, construct a sequence of "instructions" that can be used to construct the object uniquely. The number of possible *sequences* of instructions equals the number of possible complex objects.

1. Sum and product Rules.
2. Build-up counting: $\binom{n}{k}$, n-bit sequences with k 1's; goody-bags.
3. Counting one set by counting another: bijection.
4. Permutations and combinations.
5. Binomial Theorem.

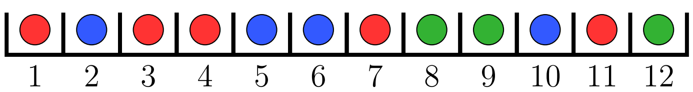
Today: Advanced Counting

1. Sequences with repetition.
   - Anagrams.
2. Inclusion-exclusion: extending the sum-rule to overlapping sets.
   - Derangements.
3. Pigeonhole principle.
   - Social twins.
   - Subset sums.

Selecting k from n Distinguishible Objects

	no repetition	with repetition
k-sequence	$\dfrac{n!}{(n-k)!}$	n^k
k-subset	$\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$	$\dbinom{k+n-1}{n-1}$
$(k_1, k_2, \cdots, k_r)$-sequence		$\dbinom{k_1 + \cdots + k_r}{k_1, k_2, \ldots, k_r} = \dfrac{k!}{k_1! \cdot k_2! \cdots k_r!}$

$(5, 4, 3)$-sequence of 5●, 4●, 3●



subset of slots used for each type

type - ●	type - ●	type - ●
$\{1, 3, 4, 7, 11\}$	$\{2, 5, 6, 10\}$	$\{8, 9, 12\}$

Choose slots for ●: $\binom{12}{5}$ ways

Then choose slots for ●: $\binom{7}{4}$ ways

Then choose slots for ●: $\binom{3}{3}$ ways

Product rule:

$$\binom{12}{5, 4, 3} = \binom{12}{5} \times \binom{7}{4} \times \binom{3}{3} = \frac{12!}{5! \cdot 7!} \times \frac{7!}{4! \cdot 3!} \times \frac{3!}{3! \cdot 0!} = \frac{12!}{5! \cdot 4! \cdot 3!}$$

Anagrams for AARDVARK

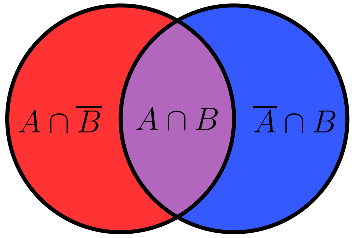
A sequence of 8 letters: 3A's, 2R's, 1D, 1V, 1K.

Number of such sequences is

$$\binom{8}{3,2,1,1,1} = \frac{8!}{3!\cdot 2!\cdot 1!\cdot 1!\cdot 1!} = 3360.$$

Exercise. What is the coefficient of $x^2y^3z^4$ in the expansion of $(x+y+z)^9$?
[Hint: Sequences of length 9 (why?) with 2 x's, 3 y's and 4 z's.]

Extending the Sum Rule to Overlapping Sets



$$|A \cup B| = |A| + |B| - |A \cap B|.$$

(Breaks $A \cup B$ into smaller sets.)

Example. How many numbers in $1, \ldots, 10$ are divisible by 2 OR 5.

$A = \{\text{numbers divisible by } 2\}$. $\quad |A| = 5$. $\quad (|A| = \lfloor 10/2 \rfloor)$

$B = \{\text{numbers divisible by } 5\}$. $\quad |B| = 2$. $\quad (|B| = \lfloor 10/5 \rfloor)$

$A \cap B = \{\text{numbers divisible by } 2 \text{ AND } 5\}$. $\quad |A \cap B| = 1$. $\quad (|A \cap B| = \lfloor 10/\mathrm{lcm}(2,5) \rfloor)$

$\boldsymbol{A \cup B = \{}$**numbers divisible by 2 or 5**$\boldsymbol{\}}$.

$$|A \cup B| = |A| + |B| - |A \cap B| = 5 + 2 - 1 = 6.$$

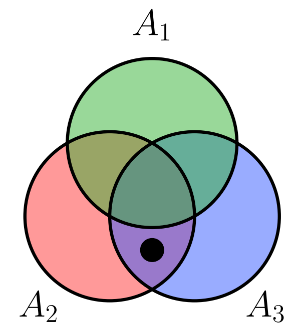
Inclusion-Exclusion

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

Proof. Consider $x \in A_2 \cap A_3$. How many times is x counted?

$\lvert A_1\rvert$	+	$\lvert A_2\rvert$	+	$\lvert A_3\rvert$	−	$\lvert A_1 \cap A_2\rvert$	−	$\lvert A_1 \cap A_3\rvert$	−	$\lvert A_2 \cap A_3\rvert$	+	$\lvert A_1 \cap A_2 \cap A_3\rvert$
0		+1		+1		0		0		−1		0

Contribution of x to sum is $+1$. Repeat for each region. ■



Example (Derangements). Give 3 coats to 3 girls so that noone gets their coat. How many ways?

$A_i = \{$girl i gets her coat$\}$. $|A_i| = 2!$.

$A_{ij} = \{$girls i and j get their coats$\}$. $|A_{ij}| = 1!$.

$A_{123} = \{$girls $1, 2$ and 3 get their coats$\}$. $|A_{123}| = 1$.

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| - |A_{12}| - |A_{13}| - |A_{23}| + |A_{123}| \\ &= 2 + 2 + 2 - 1 - 1 - 1 + 1 \;=\; 4. \end{aligned}$$

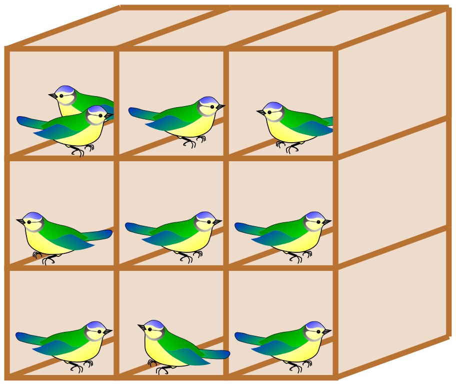
The answer we seek is $3! - 4 = 2$. (why?)

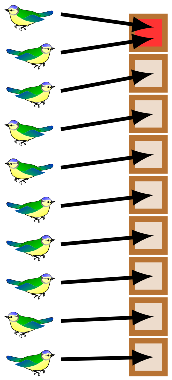
Exercise. How many numbers in 1,...,100 are divisible by 2,3 or 5?

Pigeonhole Principle

If you have more guests than spare rooms, then some guests will have to share.

- ***More* pigeons than pigeonholes.**
- **A pigeonhole has two or more pigeons.**





Proof. (By contraposition). Suppose no pigeonhole has 2 or more pigeons.
Let x_i be the number of pigeons in hole i, $x_i \leq 1$.

$$\text{number of pigeons} \;=\; \sum_i x_i \;\leq\; \sum_i 1 \;=\; \text{number of pigeonholes.} \qquad \blacksquare$$

Example. If you have 8 people, at least two are born on the same day of the week.

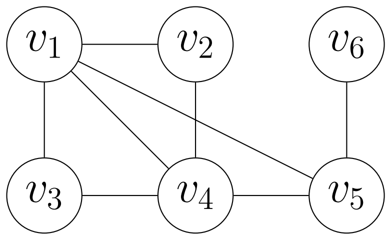
We have 8 pigeons (the people) and 7 pigeonholes (the days of the week).

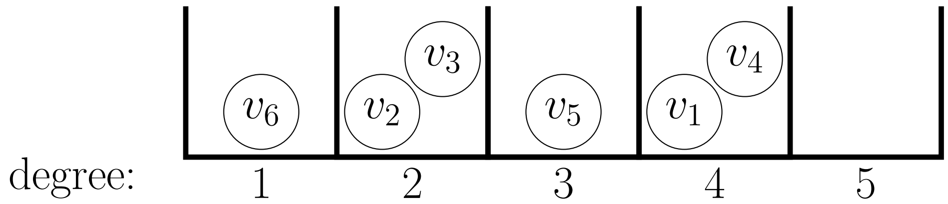
How many people do you need to ensure two are born on a Monday?

Every Graph Has At Least One Pair of Social Twins

Two nodes are *social twins* if they have the same degree.

Assume the graph is connected.





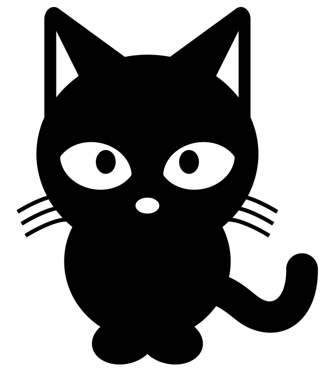
Degrees $1, 2, \ldots, (n-1)$, the pigeonholes. (Why no degree 0?)
Vertices $v_1, v_2, \ldots, v_n$, the pigeons.

n pigeons and $(n-1)$ pigeonholes, so at least two vertices are in the same degree-bin.

If the graph is not connected, no one has degree $n-1$.

Non-constructive proof: Who are those social twins? What are their degrees?

Non-Constructive Proof and the Eye-Spy Dilemma



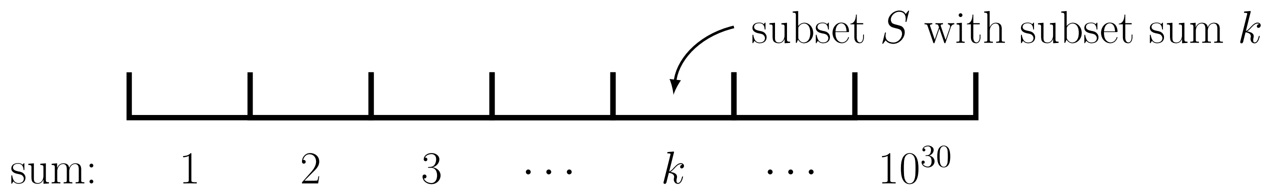
Prove to the 4 year old that the target exists in the picture

Subset Sums

Given 100 twenty-seven digit numbers, find two subsets with the same subset-sum.

Prove that the professor is not sending the student on a wild-goose chase.

Any 27-digit number is at most 10^{28}. So, a subset-sum is at most $100 \times 10^{28} = 10^{30}$.



Pigeonholes: bins corresponding to each possible subset-sum, $1, 2, \ldots, 10^{30}$.

Pigeons: the non-empty subsets of a 100-element set: $2^{100} - 1 \approx 1.26 \times 10^{30}$ of them.

At least two subsets must be in the same subset-sum-bin.

Practice. Exercise 14.6.