Foundations of Computer Science
Lecture 10

Number Theory

Division and the Greatest Common Divisor
Fundamental Theorem of Arithmetic
Cryptography and Modular Arithmetic
RSA: Public Key Cryptography

Last Time

1. Why sums and reccurrences? Running times of programs.

2. Tools for summation: constant rule, sum rule, common sums and nested sum rule.

3. Comparing functions - asymptotics: Big-Oh, Theta, Little-Oh notation.

$$\log\log(n) < \log^{\alpha}(n) < n^{\epsilon} < 2^{\delta n}$$

4. The method of integration - estimating sums.

$$\sum_{i=1}^{n} i^k \sim \frac{n^{k+1}}{k+1} \qquad \sum_{i=1}^{n} \frac{1}{i} \sim \ln n \qquad \ln n! = \sum_{i=1}^{n} \ln i \sim n\ln n - n$$

Today: Number Theory

1. Division and Greatest Common Divisor (GCD)
   - Euclid's algorithm
   - Bezout's identity
2. Fundamental Theorem of Arithmetic
3. Modular Arithmetic
   - Cryptography
   - RSA public key cryptography

The Basics

Number theory has attracted the best of the best, because

"Babies can ask questions which grown-ups can't solve" – P. Erdős

$6 = 1 + 2 + 3$ is *perfect* (equals the sum of its proper divisors). Is there an odd perfect number?

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Quotient-Remainder Theorem

For $n \in \mathbb{Z}$ and $d \in \mathbb{N}$, $n = qd + r$. The quotient $q \in \mathbb{Z}$ and remainder $0 \leq r < d$ are *unique*.

e.g. $n = 27, d = 6$: $\quad 27 = 4 \cdot 6 + 4 \quad \rightarrow \quad \text{rem}(27, 6) = 4.$

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Divisibility. d divides n, $d|n$ if and only if $n = qd$ for some $q \in \mathbb{Z}$. e.g. $6|24$.

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Division Facts (Exercise 10.2)

1. $d|0$.
2. If $d|m$ and $d'|n$, then $dd'|mn$.
3. If $d|m$ and $m|n$, then $d|n$.
4. If $d|n$ and $d|m$, then $d|n + m$.
5. If $d|n$, then $xd|xn$ for $x \in \mathbb{N}$.
6. If $d|m + n$ and $d|m$, then $d|n$.

Greatest Common Divisor

Divisors of 30: $\{1, 2, 3, 5, 6, 15, 30\}$. Divisors of 42: $\{1, 2, 3, 6, 7, 14, 21, 42\}$. Common divisors: $\{1, 2, 3, 6\}$.

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Definition. Greatest Common Divisor, GCD

Let m, n be two integers not both zero. $\gcd(m, n)$ is the largest integer that divides both m and n: $\gcd(m, n)|m$, $\gcd(m, n)|n$ and any other common divisor $d \leq \gcd(m, n)$.

Notice that every common divisor divides the GCD. Also, $\gcd(m, n) = \gcd(n, m)$.

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Relatively Prime

If $\gcd(m, n) = 1$, then m, n are relatively prime.

Example: 6 and 35 are not prime but they are relatively prime.

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$d|m$ and $d|r \to d$ divides $n = qm + r$. Hence, $d \leq \gcd(m, n) = D$. (d is a common divisor of m, n)
$D \leq d$ and $D \geq d \to D = d$, which proves $\gcd(m, n) = \gcd(n, r)$. ■

Euclid's Algorithm

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$$\gcd(42, 108) \;=\; \gcd(24, 42) \qquad 24 = \mathbf{108} - 2 \cdot \mathbf{42}$$

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$$\begin{aligned} \gcd(42, 108) &= \gcd(24, 42) & \quad 24 &= \mathbf{108} - 2 \cdot \mathbf{42} \\ &= \gcd(18, 24) & \quad 18 &= 42 - 24 = 42 - \underbrace{(108 - 2 \cdot 42)}_{24} = 3 \cdot \mathbf{42} - \mathbf{108} \end{aligned}$$

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This will be true for $\gcd(m, n)$ in general:

$$\gcd(m, n) = mx + ny \qquad \text{for some } x, y \in \mathbb{Z}.$$

Bezout's Identity: A "Formula" for GCD

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$\gcd(m, n)$ is the *smallest positive integer linear combination* of m and n:

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Formal Proof. Let ℓ be the smallest positive linear combination of m, n: $\ell = mx + ny$.

- Prove $\ell \geq \gcd(m, n)$ as above.
- Prove $\ell \leq \gcd(m, n)$ by showing ℓ is a common divisor ($\text{rem}(m, \ell) = \text{rem}(n, \ell) = 0$).

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There is no "formula" for GCD. But this is close to a "formula".

GCD Facts

ⓘ $\gcd(m, n) = \gcd(m, \mathrm{rem}(n, m))$. ✓

Proof. ∎

GCD Facts

1. $\gcd(m, n) = \gcd(m, \text{rem}(n, m))$. ✓
2. Every common divisor of m, n divides $\gcd(m, n)$.

Proof.

(e.g. 1,2,3,6 are common divisors of 30,42 and all divide the GCD 6)

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If the producers of Die Hard had chosen 3 and 6 gallon jugs, there can be no sequel (phew 🙂). (Why?)

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Theorem. Uniqueness of Prime Factorization

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Proof. (FTA) Contradiction. Let n_* be the smallest counter-example, $n_* > 2$ and

$$\begin{aligned} n_* &= p_1 p_2 \cdots p_n \\ &= q_1 q_2 \cdots q_k \end{aligned}$$

Since $p_1 | n_*$, it means $p_1 | q_1 q_2 \cdots q_k$ and by Euclid's Lemma, $p_1 = q_i$ (w.l.o.g. q_1).

$$\begin{aligned} n_*/p_1 &= p_2 \cdots p_n \\ &= q_2 \cdots q_k. \end{aligned}$$

That is, n_*/p_1 is a smaller counter-example. **FISHY!** ■

Cryptography 101: Alice and Bob wish to securely exchange the prime M

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$$M \xrightarrow{\text{Alice encrypts}} M_*$$

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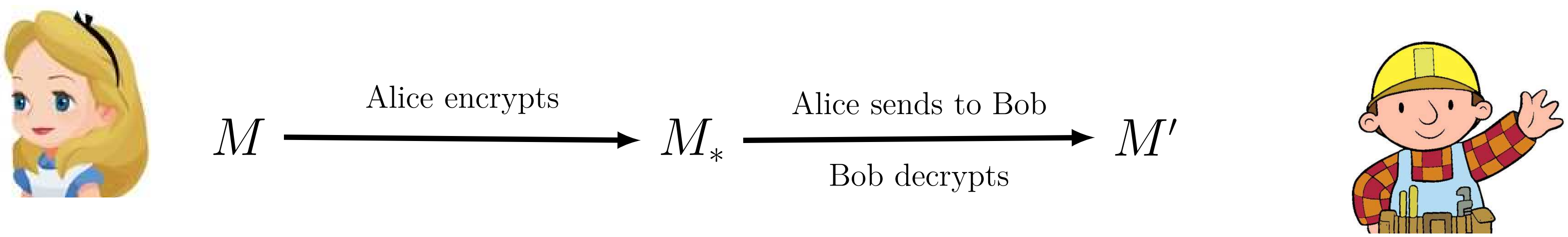

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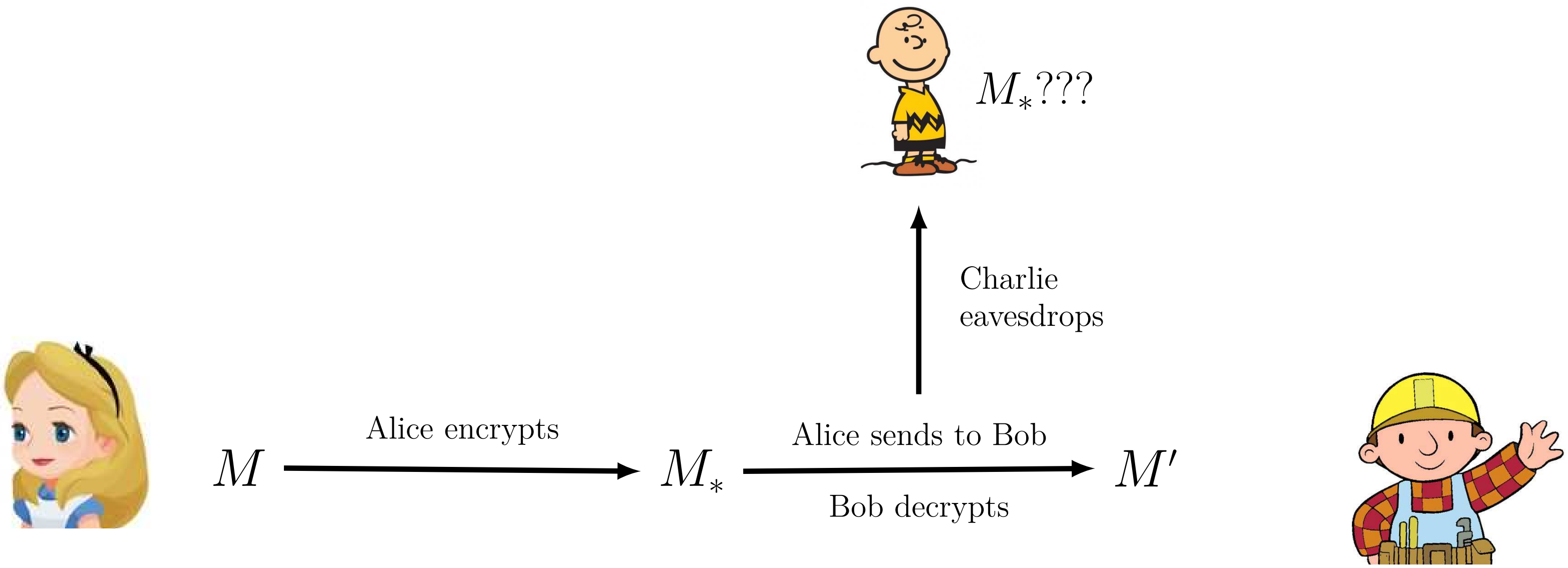

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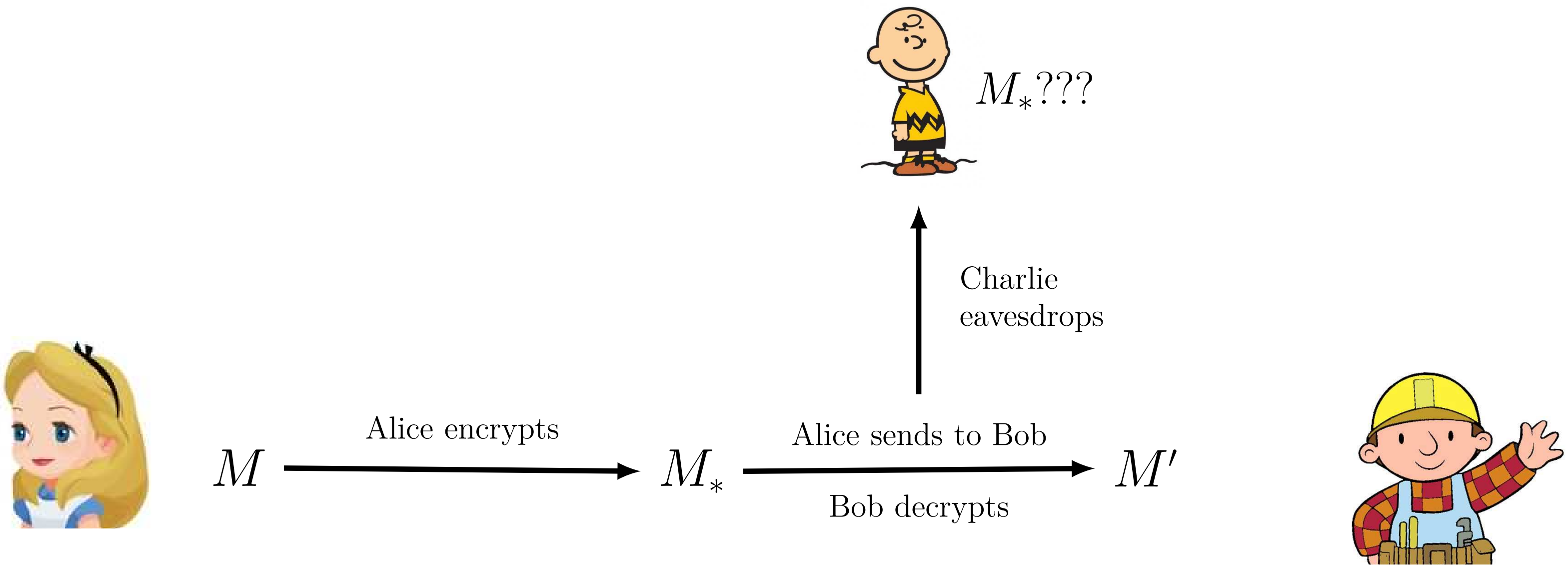

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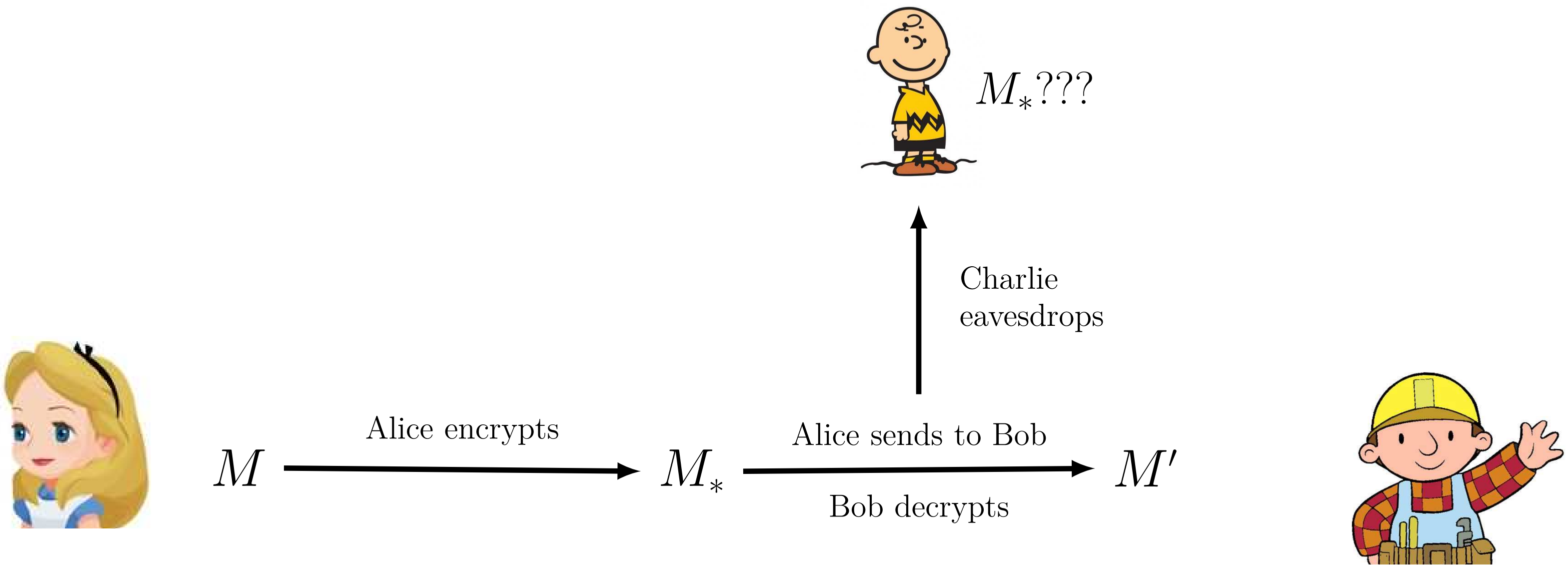

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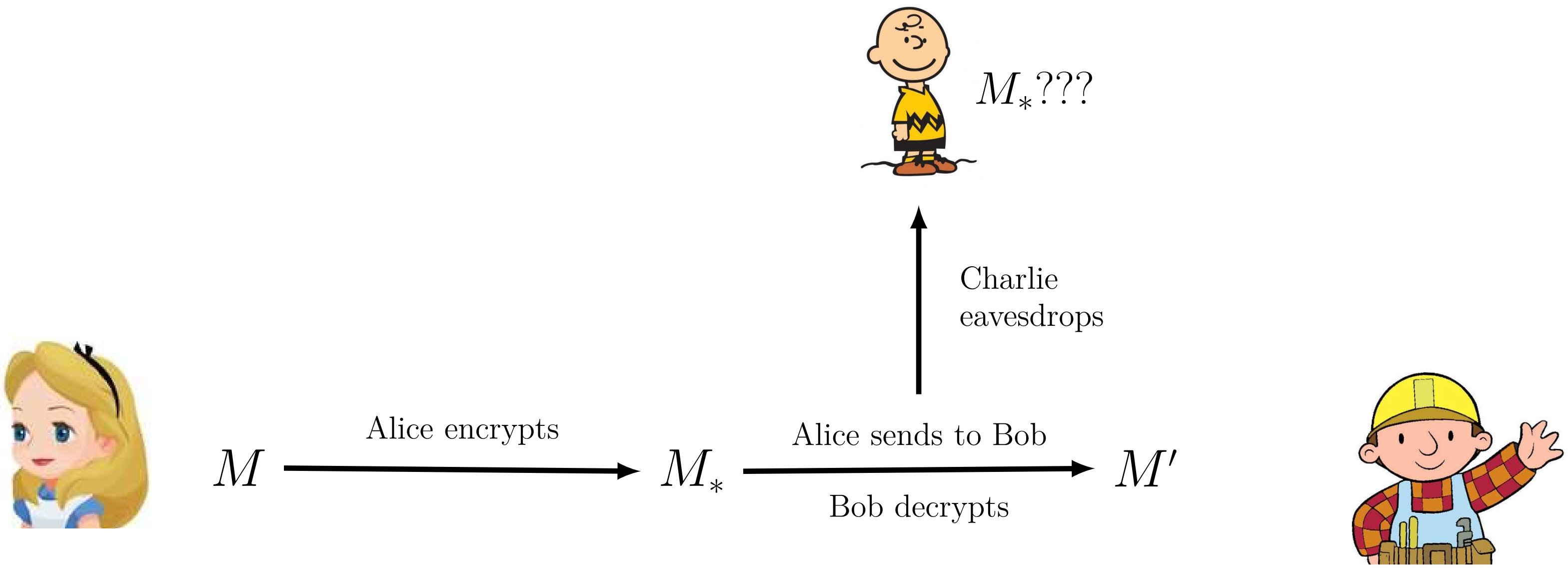

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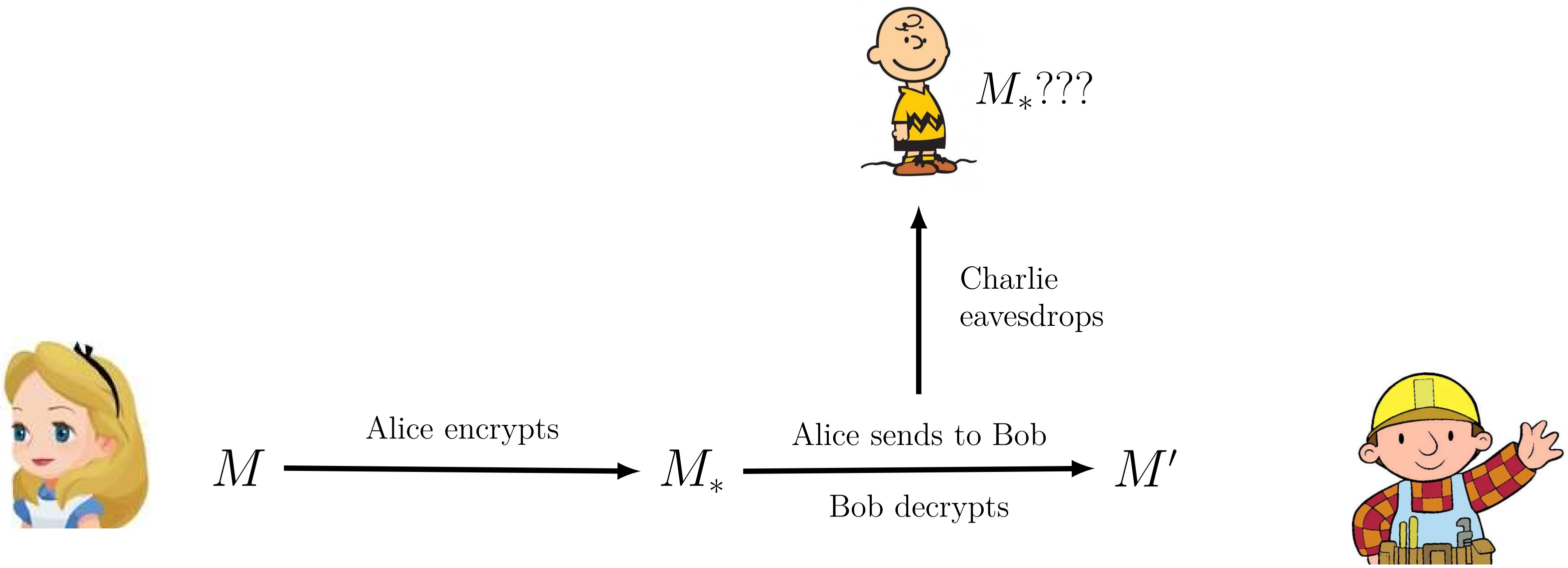

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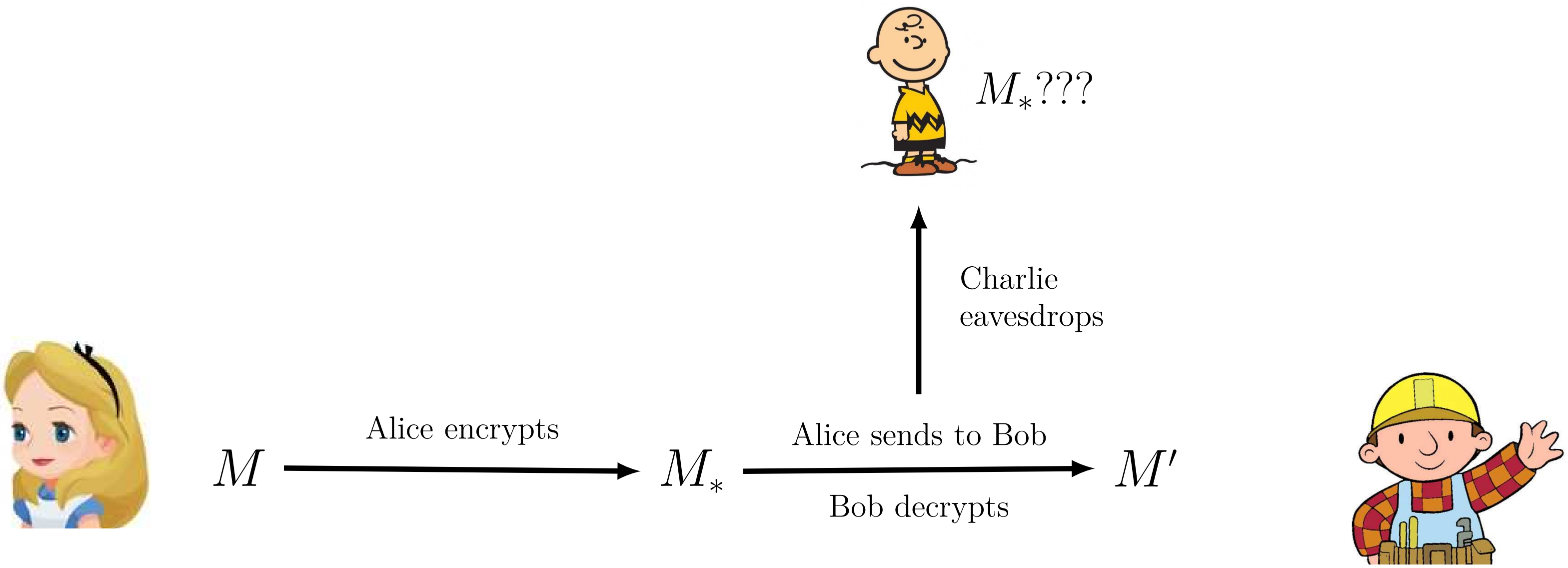

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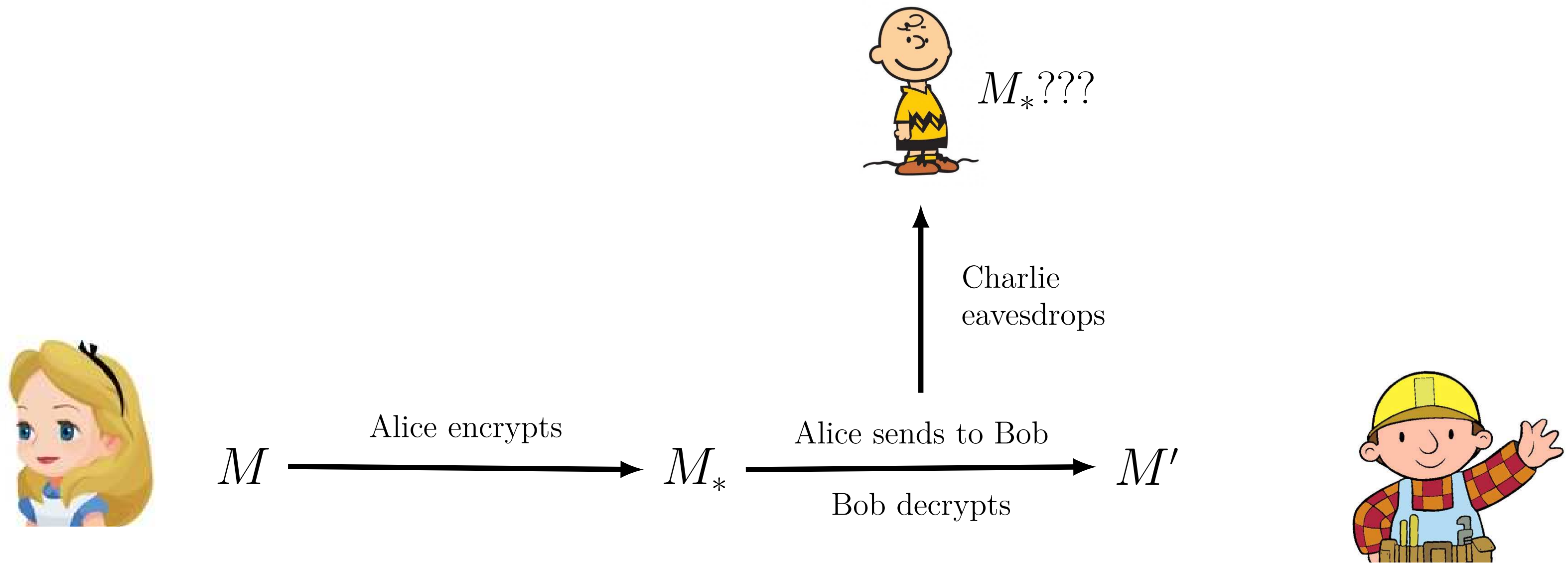

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To improve, we need modular arithmetic.

Modular Arithmetic

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Addition and multiplication are just like regular arithmetic.

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Modular Division: cancelling a factor from both sides

Suppose $ac \equiv bc \pmod{d}$. You can cancel c to get $a \equiv b \pmod{d}$ if $\gcd(c, d) = 1$.

Proof. $d|c(a-b)$. By GCD fact (v), $d|a-b$ because $\gcd(c, d) = 1$. ■

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$$3 \times n = 1 \qquad\qquad n = ?$$
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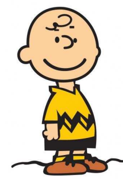

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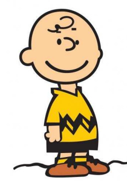

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Examples. Does Bob always decode to the correct message?

$M = 2.\quad M_* = 8 \qquad M' = 2 \qquad M' = M$

$2^{23} \equiv 8 \pmod{55} \qquad 8^7 \equiv 2 \pmod{55}$

$M = 3.\quad M_* = 27 \qquad M' = 3 \qquad M' = M$

$3^{23} \equiv 27 \pmod{55} \qquad 27^7 \equiv 3 \pmod{55}$

Exercise 10.14. Proof that Bob always decodes to the right message for special 55,23 and 7. (How to get them?)

Practical Implementation. Good idea to pad with random bits to make the cypher text random.