QUIZ 3: 120 Minutes

ISSUES

- typos in problem 13

Answer **ALL** questions.
OPEN BOOK (notes, assignments, and textbook) and electronic devices allowed.
NO COLLABORATION or Internet use. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

Circle at most one answer per question.
10 points for each correct answer.

You **MUST** show **CORRECT** work to get full credit.

When in doubt, TINKER.

Total
200

1. What is the expected number of times a six appears when a fair die is rolled ten times?

- [A] $2\frac{2}{3}$
- [B] $\frac{1}{6}$
- [C] $1\frac{2}{3}$
- [D] $1\frac{1}{3}$
- [E] None of the above

Let $x_i = \begin{cases} 1 & \text{if } 6 \text{ appears on } i\text{-th roll}, \text{ w.p. } \frac{1}{6} \\ 0 & \text{otherwise}, \text{ w.p. } \frac{5}{6} \end{cases}$

$\Rightarrow \mathbb{E}x_i = \frac{1}{6}$ for $i = 1, \dots, 10$

$\Rightarrow \mathbb{E}\left[\sum_{i=1}^{10} x_i\right] = \frac{10}{6} = 1\,2/3$

2. A test has twenty-five multiple-choice questions worth four points each and fifty True-False questions worth two points each. The probability that Katie answers a multiple choice question correctly is 0.8 and for a True-False question this probability is 0.9. What is her expected score on the test?

- [A] 200
- [B] 150
- [C] 100
- [D] 170
- [E] None of the above

$m_i = \begin{cases} 1 & \text{if answer multiple choice question } i \text{ correctly}, \text{ w.p. } \frac{8}{10} \\ 0 & \text{otherwise}, \text{ w.p. } \frac{2}{10} \end{cases}$

$t_i = \begin{cases} 1 & \text{if answer T-F question correctly}, \text{ w.p. } \frac{9}{10} \\ 0 & \text{otherwise}, \text{ w.p. } \frac{1}{10} \end{cases}$

$\Rightarrow \mathbb{E}\,\text{score} = \mathbb{E}\left[4\sum_{i=1}^{25} m_i + 2\sum_{i=1}^{50} t_i\right] = 4 \cdot 25 \cdot \frac{8}{10} + 2 \cdot 50 \cdot \frac{9}{10}$
$= 80 + 90$
$= 170$

3. We roll n fair dice. The i-th dice has x_i sides, so takes on one of the values $1, 2, \dots, x_i$. What is the expected sum of the values of these n dice?

- [A] $\frac{n}{2} + \frac{1}{2}\sum_{i=1}^{n} x_i$
- [B] $\frac{n}{2} + \sum_{i=1}^{n} x_i$
- [C] $\frac{1}{2}\sum_{i=1}^{n} x_i$
- [D] $\frac{n+1}{2}$
- [E] None of the above

r_i, outcome of rolling dice i, is uniformly dist. on $1, \dots, x_i \Rightarrow \mathbb{E}r_i = \frac{1}{x_i}(1 + \dots + x_i) = \frac{1}{x_i}\frac{x_i(x_i+1)}{2}$
$= \frac{1}{2}(x_i + 1)$

$\mathbb{E}\left[\sum_{i=1}^{n} r_i\right] = \sum_{i=1}^{n} \mathbb{E}r_i = \frac{1}{2}\sum_{i=1}^{n}(x_i + 1) = \frac{n}{2} + \frac{1}{2}\sum_{i=1} x_i$

4. X is a random variable that represents a roll of a fair six-sided die. What is the variance of X?

- [A] $\frac{7}{2}$
- [B] $\frac{71}{6}$
- [C] $\frac{49}{4}$
- [D] $\frac{91}{6}$
- [E] $\frac{35}{12}$

$\mathbb{E}X = \frac{1}{6}(1 + \dots + 6) = \frac{1}{6} \cdot \frac{6 \cdot 7}{2} = \frac{7}{2}$

$\mathrm{Var}(X) = \mathbb{E}(X - \mathbb{E}X)^2 = \frac{1}{6}\sum_{i=1}^{6}\left(i - \frac{7}{2}\right)^2 \overset{\text{algebra}}{=} \dots = \frac{35}{12}$

5. Which of the following are countable?

(I) $\mathbb{Z} \times \mathbb{Z} = \{(u, v) \mid u \in \mathbb{Z} \text{ and } v \in \mathbb{Z}\}$
(II) The set of unrecognizable languages
(III) The set of solvable problems

- [A] I & III
- [B] I only

(I) is countable b/c of the Cantor diagonalization argumen

(III) is the set of Turing deciders, which is contained in the set of Turing machines, which is countable, so (III) is countable

(II) the set of recognizable languages is countable by the same argument for (III), and the set of all languages is uncountable, so the set of unrecognizable languages is uncountable

C II & III

D I & II

E I, II, and III

6. Which of the following strings match the regular expression $\{0, 01\}^* \bullet \{1, 10\}^*$?

(I) 101110 (II) 00111 (III) 00100 (IV) 01100

(I) and (III) are in $\{0,01\}^*$

(II) is in the language because $00 \in \{0,01\}^*$ and $111 \in \{1,10\}^*$

A II and IV

B III

C all except IV

D all except I

E all

7. What is the correct relationship between the cardinalities of these sets:

(I) $\mathcal{A}$, the set of all languages ← uncountable

(II) $\mathcal{I}$, the interval $[0, 1]$ ← uncountable

(III) $\mathcal{C}$, the set of C programs that compile successfully and halt eventually when run ← countable

A $|\mathcal{C}| = |\mathcal{A}| < |\mathcal{I}|$

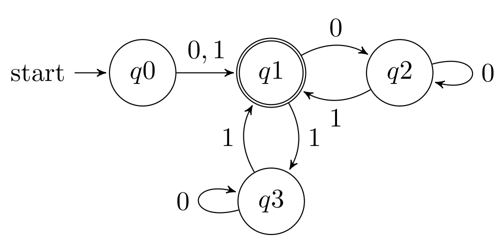
B $|\mathcal{C}| = |\mathcal{A}| = |\mathcal{I}|$

C $|\mathcal{I}| < |\mathcal{C}| = |\mathcal{A}|$

D $|\mathcal{A}| = |\mathcal{I}| \leq |\mathcal{C}|$

E $|\mathcal{C}| < |\mathcal{A}| = |\mathcal{I}|$

8. Consider the following DFA. Which of these strings will it accept: (I) 011011 (II) 100110 (III) 111101 ?



A I & II

B II & III

C III only

D I only

E none

9. Which is the following claims is true about the language $\mathcal{L} = \{\omega\#\omega^R\#\omega \mid w \in \{0,1\}^*\}$?

A Its complement is regular

B It is not decidable but is recognizable

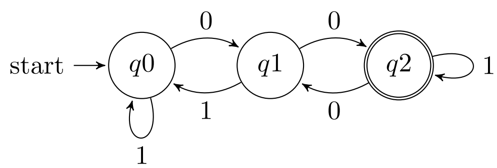
C It is context-free

D It is not context-free but is decidable

E It can be recognized with a PDA

(requires random access to a memory to solve)

10. Which of the following languages will *not* be accepted by this DFA?



A $\{00\} \bullet \{1\}^*$

B $\{00\} \bullet \{1^*00\}^*$

C $\{0\} \bullet \{1\}^* \bullet \{0\} \bullet \{0\}^* \bullet \{1\}$

D $\{100\} \bullet \{100\}^*$

E $\{0\} \bullet \{10\}^* \bullet \{01\}$

e.g. 0101 is in this language but not accepted by the DFA

11. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are both undecidable but recognizable languages, which of the following are also recognizable: (I) $\overline{\mathcal{L}_1}$ (II) $\mathcal{L}_1 \cap \mathcal{L}_2$ (III) $\mathcal{L}_1 \cup \mathcal{L}_2$

Hint: Given recognizers for $\mathcal{L}_1$ and $\mathcal{L}_2$, how could you build recognizers for these languages?

A I

B I and II

C II

D II and III

E III

– for (II), run recognizers for $\mathcal{L}_1$ and $\mathcal{L}_2$ simultaneously, both will accept if $w \in \mathcal{L}_1 \cap \mathcal{L}_2$ so this is a recognizer

– for (III), do the same, and at least one will accept if $w \in \mathcal{L}_1 \cup \mathcal{L}_2$, in that case ACCEPT $\Rightarrow$ this is a recognizer for $\mathcal{L}_1 \cup \mathcal{L}_2$

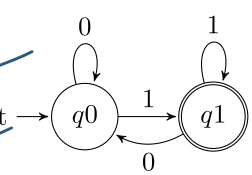
– for (I), if $\overline{\mathcal{L}_1}$ were recognizable and $\mathcal{L}_1$ were as well, we could build a decider for both languages, so conclude $\overline{\mathcal{L}_1}$ is not recognizable

12. How many strings of length four are accepted by this DFA?

Enumerate and check:

0000 x
0001 ✓
0010 x
0011 ✓
0100 x
0101 ✓
0110 x
0111 ✓

1000 x
1001 ✓
1010 x
1011 ✓
1100 x
1101 ✓
1110 x
1111 ✓



A 5

B 6

C 8

D 10

E 12

FREE POINTS DUE TO TYPO

13. Generate a random two digit binary string by choosing each digit independently and identically, selecting zero with probability 1/3 and one with probability 2/3. What is the probability that the automaton from the previous problem will accept a string generated in this manner?

A $\frac{2}{9}$

B $\frac{4}{9}$

C $\frac{5}{9}$

D $\frac{1}{3}$

E $\frac{7}{9}$

this should have been 6/9

00 ✗
01 ✓
10 ✗
11 ✓

$\mathbb{P}(\text{accept}) = \mathbb{P}(01 \text{ or } 10)$
$= \mathbb{P}(01) + \mathbb{P}(11)$
$= \frac{1}{3} \cdot \frac{2}{2} + \left(\frac{2}{3}\right)^2 = \frac{2}{9} + \frac{4}{9} = \frac{6}{9}$

14. If the complement of a language is countable, which of the following are necessarily true: (I) the language is regular (II) the language is decidable (III) the language is context-free

A all

B none

C II only

D I only

E III only

all languages are countable, so $\overline{\mathcal{L}}$ being countable tells us nothing about $\mathcal{L}$. $\mathcal{L}$ could have all or none of properties (I)–(III)

15. Describe the language generated by this CFG.

1: $S \to A1B$
2: $A \to \varepsilon|0A$ ← generates $\{0\}^*$
3: $B \to \varepsilon|0B|1B$ ← generates $\{0,1\}^*$

⇒ CFG generates $\{0\}^* 1 \{0,1\}^*$

A The set of strings that starts with zero and contains a one

B The set of strings with an odd number of zeros

C The set of strings containing a one

D The set of strings with more ones than zeros

E None of the above

16. Consider the CFG

1: $S \to 0|SA$
2: $A \to AA|S1$

Which string is in the language described by this CFG?

all strings will start with zero. (eliminates A & D)

A 10101

B 001

C 011

D 101

E None of the above

enumerate the strings it generates

0, 001, 0001

↑ only string of length 3 (eliminates C)

17. Which of the following CFGs generates all finite binary strings?

(I) $S \to \varepsilon|0S|1S$

(II) $S \to \varepsilon|1|0S|S1$
(III) $S \to \varepsilon|0|S1|SS$

$\boxed{\text{A}}$ I and II

$\boxed{\text{B}}$ II and III

$\boxed{\text{C}}$ I and III

$\boxed{\text{D}}$ I

$\boxed{\text{E}}$ all three

18. If $\mathcal{L}$ is undecidable, which of the following *cannot* be true?

$\boxed{\text{A}}$ There is a recognizer for $\mathcal{L}$

$\boxed{\text{B}}$ $\mathcal{L} \subseteq \mathcal{L}_{\text{HALT}}$

$\boxed{\text{C}}$ $\overline{\mathcal{L}}$ is decidable

$\boxed{\text{D}}$ $\mathcal{L}$ is countable

$\boxed{\text{E}}$ Any of the above could be true

if $\overline{\mathcal{L}}$ were decidable, we can get a decider for $\mathcal{L}$ by inverting the output of the decider for $\overline{\mathcal{L}}$. This contradicts the undecidability of $\mathcal{L}$, so $\overline{\mathcal{L}}$ must also be undecidable.

19. Which CFG generates the same language as

1: $S \to 00S1$
2: $T \to 0S1$
3: $S \to 0T$
4: $S \to \varepsilon|01$

} equivalent to $S \to 00S1$

$\boxed{\text{A}}$ $S \to \varepsilon|01|00S11$

$\boxed{\text{B}}$ $S \to \varepsilon|01|0S1|00S1$

$\boxed{\text{C}}$ $S \to \varepsilon|01|00S1$

$\boxed{\text{D}}$ $S \to \varepsilon|01|0000S11$

$\boxed{\text{E}}$ $S \to \varepsilon|01|000S1$

20. Under which of the following operations is the class of decidable problems closed: (I) complementation (II) union (III) intersection (IV) Kleene-Star ?

Hint: how would you construct deciders for languages defined using these operations?

$\boxed{\text{A}}$ all except IV

$\boxed{\text{B}}$ II and III

$\boxed{\text{C}}$ all except I

$\boxed{\text{D}}$ all

$\boxed{\text{E}}$ none

(I) get a decider for $\overline{\mathcal{L}}$ by inverting the output of the decider for $\mathcal{L}$

(II) ACCEPT if either $\mathcal{L}_1$ or $\mathcal{L}_2$ deciders accept, REJECT otherwise

(III) ACCEPT if both $\mathcal{L}_1$ and $\mathcal{L}_2$ decide accept, REJECT otherwise

(IV) write an algorithm that generates all partitions of w into substrings and runs the decider for $\mathcal{L}$ on each part of the partitions. ACCEPT if there's a partition whose parts are all in $\mathcal{L}$. Else REJECT