MIDTERM: 120 Minutes

Last Name: ________________________

First Name: ________________________

RIN: ________________________

Section: ________________________

Answer **ALL** questions. You may use one double sided $8\frac{1}{2} \times 11$ crib sheet.

NO COLLABORATION or electronic devices. Any violations result in an F.

NO questions allowed during the test. Interpret and do the best you can.

You **MUST** show **CORRECT** work, **even on multiple choice questions**, to get credit.

GOOD LUCK!

1	2	3	4	5	6	Total
100	20	20	20	20	20	200

1 Circle one answer per question. 10 points for each correct answer.

(a) Compute the sum $\sum_{n=1}^{4} 5^{n+1}$.

$$\sum_{n=1}^{4} 5^{n+1} = \sum_{n=2}^{5} 5^n = \sum_{n=0}^{5} 5^n - 1 - 5$$

$$= \frac{1-5^6}{1-5} - 6 = 3900$$

↑ common sum

A 3900.

B 3901.

C 3905.

D 3906.

E None of the above.

(b) Compute $20^{20} \bmod 7$

$$20^{20} \bmod 7 = (-1)^{20} \bmod 7 = 1$$

A 1.

B 3.

C 4.

D 6.

E None of the above.

(c) A graph has degree sequence $[6, 6, 4, 3, 3, 2, 2]$. How many edges does this graph have?

$$\sum s_i = 2|E| = |E| = 13$$

A 13.

B 25.

C 30.

D Not enough information to say.

E Such a graph does not exist.

(d) Suppose a connected planar graph has 4 vertices and splits the plane into 3 regions. Which of the following are possible degree sequences for the graph?

$$F + V - E = 2 \Rightarrow E = 5$$

$$\sum s_i = 2E \Rightarrow \sum s_i = 10$$

A [2 2 2 2].

B [3 3 3 3].

C [3 3 2 2].

D None of the above.

E No such graph exists.

(e) What is the last digit of 103^{192}.

$$103^{192} \bmod 10 = 3^{192} \bmod 10 = 9^{96} \bmod 10 = (-1)^{96} \bmod 10$$

$$= 1$$

A 0

B 1

C 2

D 3

E 4

(f) Which of the following numbers evenly divides $5^{69} - 1$?

A 4

B 5

C 11

D 23

E None of the above

$5^{69} \bmod 4 = 1^{69} \bmod 4 = 1$

$\Rightarrow (5^{69} - 1) \bmod 4 = 0$

(g) The converse of "If induction is appropriate then the base case is true and the inductive step holds" is:

A If the base case is false and the inductive step is false, then induction is not appropriate.

B If the base case is false or the inductive step does not hold, then induction is not appropriate.

C If induction is not appropriate, then the base case is false or the inductive step does not hold.

D If the base case is true and the inductive step holds then induction is appropriate.

E None of the above.

converse of $p \to q$ is $q \to p$

(h) Which claim below is *not* true?

A $2n^2 + n \in \Theta(n^2)$.

B $4^n \in \Theta(2^n)$.

C $f \in \Theta(n)$ and $g \in \Theta(n) \Rightarrow f + g \in \Theta(n)$.

D None of these claims are true.

E All of these claims are true.

$\frac{4^n}{2^n} = 2^n \to \infty$ as $n \to \infty$

so $4^n \notin \Theta(2^n)$

(i) Suppose $f(x) > 0$ for all x, and $f(i+1)/f(i) \leq r$, where $0 < r < 1$. For which of the following g is $\sum_{i=1}^{n} f(i) \in \Theta(g(n))$?

A $g(n) = 1$.

B $g(n) = 2^r$.

C $g(n) = \ln(r)$.

D $g(n) = r^n$.

E None of the above.

$f(i+1) \leq f(i) r \leq \cdots \leq f(1) r^i$

$\Rightarrow f(1) \leq \sum_{i=1}^{n} f(i) \leq f(1) \sum_{i=1}^{n} r^{i-1} = f(1) \frac{1-r^n}{1-r}$

so $\sum_{i=1}^{n} f(i) \in \Theta(f(1)) = \Theta(1)$

(j) Which of these sums are $O(n^2)$: (a) $\sum_{i=1}^{n}(1+i)^2$ (b) $\sum_{i=1}^{n} 2^i$ (c) $\sum_{i=1}^{n} \frac{i}{1+i^2}$ (d) $\sum_{i=1}^{n}(-1)^i i$?

$= \Theta(n^3)$ $= \Theta(2^n)$ $= \leq \sum i = \Theta(n^2)$ $\sum i = \Theta(n^2)$

A a, b

B c

C a, b, c

D a, c

E c, d

2 **Let $d = \gcd(m,n)$, where $m,n > 0$. Bezout's Theorem gives $d = mx + ny$ where $x, y \in \mathbb{Z}$. Prove or disprove that it is always possible to find $a, b \in \mathbb{Z}$ for which $ax + by = 1$.**

This is true. In other words, we can express the gcd of m, n as $d = mx + ny$ where $\gcd(x,y) = 1$.

<u>Prf</u> (Contradiction)

Assume that $d = mx + ny$ where $\gcd(x,y) = c > 1$.

Then

$$\frac{d}{c} = m\left(\frac{x}{c}\right) + n\left(\frac{y}{c}\right).$$

Since $\frac{x}{c}$ and $\frac{y}{c}$ are integers, as is $\frac{d}{c}$, we see that $\frac{d}{c}$ is a positive integral combination of m and n that is smaller than d. This contradicts Bezout's theorem which states d is the smallest positive integral combination of m and n. We conclude that in fact $\gcd(x,y) = 1$, so there exist $a, b \in \mathbb{Z}$ such that

$$ax + by = 1,$$

by Bezout's Theorem

3 **A leaf is a vertex with degree 1. Let Δ denote the maximum degree in a tree T. Use the hand-shaking theorem to prove that T has at least Δ leaves.**

Proof

Let n be the number of vertices in T, so T has $n-1$ edges. The handshaking theorem guarantees

$$\sum_{i=1}^{n} \delta_i = 2(n-1).$$

Let ℓ be the number of leaf vertices in T. They contribute ℓ to the sum of degrees. At least one vertex contributes Δ to the sum of degrees. All remaining vertices each contribute at least 2 to the sum of degrees. More concisely,

$$\Delta + 2(n-\ell-1) + \ell \leq \sum_{i=1}^{n} \delta_i = 2(n-1).$$

Simplifying,

$$\Delta + 2(n-1) - \ell \leq 2(n-1)$$

$$\Rightarrow \quad \Delta \leq \ell$$

as claimed.

4 **Prove, or disprove:** $n! \in \Theta(2^n)$.

This is false.

Proof

$$\frac{n!}{2^n} = \frac{1}{2}\cdot\frac{2}{2}\cdot\frac{3}{2}\cdot\dots\cdot\frac{n}{2} \geq \frac{1}{2}\cdot\underbrace{\frac{3}{2}\cdot\dots\cdot\frac{3}{2}}_{n-2\text{ terms}}$$

$$= \frac{1}{2}\left(\frac{3}{2}\right)^{n-2}$$

so $\frac{n!}{2^n} \to \infty$ as $n \to \infty$

thus $n! \in \omega(2^n)$. In particular, $n! \notin \Theta(2^n)$

$\blacksquare$

5 **For $k \in \mathbb{N}$, show that $2^k + 1$ and $2^k - 1$ are relatively prime.**

<u>Prf</u>

Recall Bezout's theorem says $\gcd(2^k+1, 2^k-1)$ is the smallest positive integral combination of 2^k+1 and 2^k-1.

By construction,

$$2 = (2^k+1) - (2^k-1)$$

is a positive integral combination of 2^k+1 and 2^k-1. This means either 1 or 2 is the desired gcd.

Since 2^k+1 and 2^k-1 are both odd numbers, 2 cannot be their gcd. We conclude that 1 must be their gcd.

$\blacksquare$

6 **Let $n \geq 1$ be a natural number. Prove that $2^{(1/2)^n}$ is not rational.**

Prf

We use induction on n.

Base case when $n=1$, we know $\sqrt{2}$ is irrational.

Induction step. Assume $2^{(1/2)^n}$ is irrational.

To see that $2^{(1/2)^{n+1}}$ is also irrational, we use contradiction. That is, assume $2^{(1/2)^{n+1}}$ is rational,

$$2^{\left(\frac{1}{2}\right)^{n+1}} = \frac{p}{q} \quad \text{for } p \in \mathbb{Z},\ q \in \mathbb{N}.$$

Squaring both sides gives

$$2^{(1/2)^n} = \frac{p^2}{q^2} \quad \text{where } p^2 \in \mathbb{Z},\ q \in \mathbb{N};$$

that is, $2^{(1/2)^n}$ is rational.

This contradicts our inductive hypothesis, so it must be the case that, in fact, $2^{\left(\frac{1}{2}\right)^{n+1}}$ is irrational.

By the Principle of Induction, it follows that $2^{\left(\frac{1}{2}\right)^n}$ is irrational for all $n \geq 1$.

$\blacksquare$

SCRATCH

SCRATCH